IDEALS, CURVES AND BOXES

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Junior Math Days

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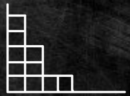
Overview

§0. Algebraic Geometry in 1 slide

§1. Abstract varieties: the example of curves

§2. Embedded varieties: the example of fat points



ALGEBRAIC GEOMETRY

∪

ENUMERATIVE GEOMETRY has interesting intersections with

touched upon today

- MATHEMATICAL PHYSICS
- STRING THEORY
- DIFFERENTIAL EQUATIONS
- NUMBER THEORY
- COMBINATORICS
- REPRESENTATION THEORY
- MOTIVIC THEORIES
- DG ALGEBRAS

[...]

construct moduli spaces and attach invariants to them: we will see

§1 the moduli space of curves, and

§2 the Hilbert scheme of points.

ALGEBRAIC GEOMETRY STUDIES ALGEBRAIC VARIETIES

AFFINE: zero loci of polynomials $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$

$$\rightsquigarrow \quad V(f_1, \dots, f_r) \subset \mathbb{A}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{C}\}$$

PROJECTIVE: zero loci of homogeneous polynomials $\bar{f}_1, \dots, \bar{f}_r \in \mathbb{C}[x_0, x_1, \dots, x_n]$

$$\rightsquigarrow \quad V(\bar{f}_1, \dots, \bar{f}_r) \subset \mathbb{P}^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^\times$$

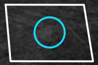
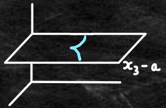
POLYNOMIALS ARE EASY, THEIR VANISHING LOCI ARE NOT!

$\deg f_i = 1 \rightsquigarrow$ LINEAR ALGEBRA :)

$n = 1$, $\deg f \leq 4 \rightsquigarrow$ SOLVE BY RADICALS :)

EVERYTHING ELSE IS NOT SO EASY!

Algebra = Geometry

polynomial rings $\mathbb{C}[x_1, \ldots, x_n]$ $\cup$ ideals I	affine spaces $\mathbb{A}^n$ $\cup$ "subvarieties" $V(I) = \{p \mid f(p)=0,\ f \in I\}$
$\langle x-a \rangle \subset \mathbb{C}[x]$	 $\mathbb{A}^1$
$\langle x_1^2+x_2^2-1 \rangle \subset \mathbb{C}[x_1,x_2]$	 $\mathbb{A}^2$
$\langle x_3-a,\ x_1^3-x_2^2 \rangle \subset \mathbb{C}[x_1,x_2,x_3]$	 $\subset \mathbb{A}^3$

§ 1. ABSTRACT VARIETIES

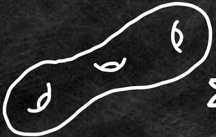
YOU ARE NOT FORCED TO REMEMBER THE EMBEDDING OF YOUR VARIETY IN $\mathbb{A}^n$, $\mathbb{P}^n$, OR ... ANY AMBIENT SPACE.

FORGET ALL EMBEDDINGS : study of "abstract varieties"

We will now see the example of curves

(i.e. $\dim = 1$, $\dim_{\mathbb{R}} = 2$)

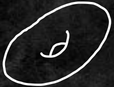
C^∞: we have Riemann surfaces



Σ_g

g = # holes, the unique C^∞ invariant.

However, in the algebraic / holomorphic world, there are many non-isomorphic structures on Σ_g.

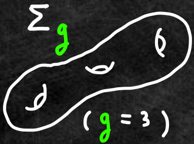


$g = 1$

can be embedded in $\mathbb{P}^2$ as $y^2 = x(x-1)(x-\lambda)$

this is one 'modulus' (parameter)

How many moduli of curves? ($g \geq 2$)

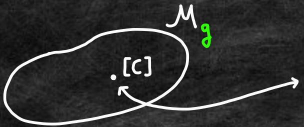


has only 1 diffeomorphism type.

But what is the dimension of the family of complex structures on Σ_g?

$\boxed{\text{Riemann's answer (1857)}: \ 3g - 3}$

TODAY WE KNOW THERE IS A VARIETY OF MODULI



smooth projective curve C of genus g (≥ 2)

$$\dim \mathcal{M}_g = 3g - 3.$$

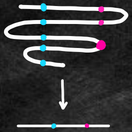
Riemann did not know this and yet "he computed $\dim \mathcal{M}_g$".

RIEMANN MODULI COUNT

LINK WITH NUMBER THEORY

(1) Fix C of genus $g \geq 2$, and fix $d > 2g-2$. Look at $\boxed{\text{COVERS}}$

$$\begin{tikzcd} C \arrow[d, "f", "d:1"'] \\ \mathbb{P}^1 \end{tikzcd}$$



$b := \left|\{z \in \mathbb{P}^1 : |f^{-1}(z)| < d\}\right|$

$\boxed{b = 2g-2+2d}$ RIEMANN-HURWITZ FORMULA

(2) b = # MODULI OF PAIRS $\left(C, C \xrightarrow{d:1} \mathbb{P}^1\right)$

RIEMANN EXISTENCE THEOREM

$\overset{(1)}{\Longrightarrow}$ $2g-2+2d =$ "$\dim M_g$" + # MODULI OF $C \xrightarrow{d:1} \mathbb{P}^1$

(3) $C \xrightarrow{d:1} \mathbb{P}^1$ is determined by 2 independent sections (up to scaling) $s, t \in \Gamma(L)$ for L a degree d line bundle on C. RIEMANN-ROCH THEOREM says

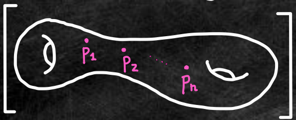
$$\dim_{\mathbb{C}} \Gamma(L) = d + 1 - g \qquad (\text{using } d > 2g-2)$$

So # MODULI OF $C \xrightarrow{d:1} \mathbb{P}^1$ is $g + 2(d+1-g) - 1$

(4) Conclude: $2g-2+2d =$ "$\dim \mathcal{M}_g$" $+ g + 2(d+1-g) - 1$

i.e. "$\dim \mathcal{M}_g$" $= 3g - 3$ $\blacksquare$

More generally, there are moduli spaces of "MARKED CURVES"

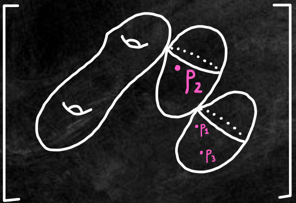


$\in \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ ← DELIGNE-MUMFORD COMPACTIFICATION

dim $= 3g - 3 + n$

SOME SINGULAR CURVES MUST BE ALLOWED IN ORDER TO GET A <u>COMPACT SPACE</u>

$\Downarrow$



$\in \overline{\mathcal{M}}_{2,3}$

$$\boxed{\text{So intersection numbers } \int_{\overline{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q} \text{ make sense!}}$$

$T^*_{p_i}C \hookrightarrow \mathbb{L}_i$ COTANGENT LINE BUNDLES $(1 \le i \le n)$

$\downarrow \qquad\qquad \downarrow$

[] $\in \overline{\mathcal{M}}_{g,n}$

$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

"GROMOV-WITTEN POTENTIAL OF A POINT"

$$F = \sum_g \sum_n \frac{1}{n!} \sum_{a_1,\dots,a_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \right) t_{a_1} \cdots t_{a_n}$$

"DESCENDENT INTEGRAL"

WITTEN'S CONJECTURE (KONTSEVICH'S THEOREM) ~1991

$\mathcal{U} := \dfrac{\partial^2 F}{\partial t_0^2}$ OBEYS THE KdV EQUATIONS (infinite series of differential equations), and F satisfies the STRING EQUATION

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i \ge 0} t_{i+1} \frac{\partial F}{\partial t_i}$$

§ 2. Embedded varieties

Fix ambient space $\mathbb{A}^2$. Look at subvarieties $V \subset \mathbb{A}^2$, $\boxed{\dim V = 0.}$

$V \subset \mathbb{A}^2 \longleftrightarrow I \subset \mathbb{C}[x,y]$. $\boxed{\mathbb{C}[x,y]/I \text{ finite dim } \mathbb{C}\text{-vector space.}}$

FIX THIS DIMENSION TO BE $n \geq 0$

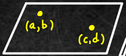
"HILBERT SCHEME OF POINTS"

$$\mathrm{Hilb}^n(\mathbb{A}^2) = \{ I \subset \mathbb{C}[x,y] \mid \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \}$$

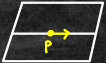
HAS A VARIETY STRUCTURE!

e.g. $n = 1$: $\mathbb{C}[x,y]/I \cong \mathbb{C} \iff I = (x-a, y-b)$, so $\mathrm{Hilb}^1(\mathbb{A}^2) \cong \mathbb{A}^2$

$$\begin{array}{ccc} \mathrm{Hilb}^1(\mathbb{A}^2) & \cong & \mathbb{A}^2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x-a, y-b) & \longleftrightarrow & (a,b) \end{array}$$

$n = 2$: either 

$\mathbb{A}^2 \longleftrightarrow (x-a,\, y-b)\cdot(x-c,\, y-d)$

or $\mathbb{C}[x,y]/I \cong \mathbb{C}[x,y]/(x^2,y)$ 

point $p \in \mathbb{A}^2$ plus infinitesimal direction

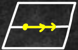
$$\mathrm{Hilb}^2(\mathbb{A}^2) = \{(p,q) \mid p \neq q\}/S_2 \;\sqcup\; \mathbb{A}^2 \times \mathbb{P}^1 \qquad (\dim = 4)$$

$n = 3$: 4 geometric situations

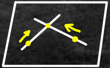
"FAT POINTS"

3 DISTINCT POINTS

2 DISTINCT POINTS



$\mathbb{C}[x,y]/(y,x^3)$



$\mathbb{C}[x,y]/(x^2,xy,y^2)$

Now forget about $V \subset \mathbb{A}^2$. Look at **PARTITIONS** of n :

$n = 1$

$1 = 1 + 0$

$n = 2$

$2 = 1 + 1$, $2 = 2 + 0$

$n = 3$

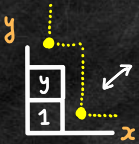
$3 = 1 + 1 + 1$, $3 = 2 + 1$, $3 = 3 + 0$

$n = 4$

$4 = 1+1+1+1$, $4 = 3 + 1$, $4 = 2 + 2$, $4 = 2 + 1 + 1$, $4 = 4 + 0$

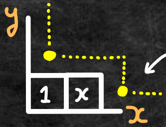
... Take a closer look: put coordinates "x, y" in the Young diagrams

and ... LOOK AT THE STAIRCASE !



IDEAL

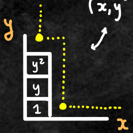
$I = (x, y^2)$



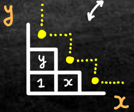
IDEAL

$I = (y, x^2)$

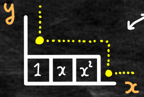
THE STAIRCASE DEFINES A MONOMIAL IDEAL $I \in \mathrm{Hilb}^{\#\text{boxes}}(\mathbb{A}^2)$



(x, y^3)



(x^2, xy, y^2)



(y, x^3)

$\longrightarrow \{\text{PARTITIONS OF } n\} \xleftrightarrow{1:1} \{I \in \mathrm{Hilb}^n(\mathbb{A}^2) \mid I \text{ MONOMIAL IDEAL}\}$

DEEP FACT:

$$\chi(\mathrm{Hilb}^n(\mathbb{A}^2)) = \left|\{I \in \mathrm{Hilb}^n(\mathbb{A}^2) \mid I \text{ MONOMIAL}\}\right|$$

$$= p(n) = \text{number of partitions of } n$$

Let's compute $Z_{\mathbb{A}^2}(t) = \sum_{n \geq 0} \chi(\mathrm{Hilb}^n(\mathbb{A}^2))\, t^n = \sum_{n \geq 0} p(n)\, t^n$

Try to expand $(1-t^{1})^{-1}(1-t^{2})^{-1}(1-t^{3})^{-1}(1-t^{4})^{-1}\cdots\cdots$

You get: $(1+t^{1}+t^{2}+t^{3}+t^{4}+t^{5}+\dots)(1+t^{2}+t^{4}+t^{6}+\dots)(1+t^{3}+t^{6}+\dots)(1+t^{4}+t^{8}+\dots)\cdots\cdots$

Say you want the coefficient of t^{4}:

LINK WITH COMBINATORICS...

1+1+1+1 2+1+1 1+3

$$(1+t^{1}+t^{1+1}+t^{1+1+1}+t^{1+1+1+1}+\dots)(1+t^{2}+t^{2+2}+\dots)(1+t^{3}+t^{3+3}+\dots)(1+t^{4}+t^{4+4}+\dots)\cdots$$

2+2 4+0

$\rightsquigarrow$ recover the 5 partitions of 4 $\Rightarrow$

EULER'S FORMULA:

$$\sum_{n\geq 0} p(n)\, t^{n} = \prod_{k\geq 1} (1-t^{k})^{-1}$$

So we have proved:

$$Z_{\mathbb{A}^2}(t) = \sum_{n \geq 0} \chi(\mathrm{Hilb}^n(\mathbb{A}^2))\, t^n = \prod_{k \geq 1} (1-t^k)^{-1}$$

$\llcorner$ LINK WITH

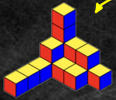
- NUMBER THEORY: $t \cdot \prod_{k \geq 1} (1-t^k)^{24}$ is a MODULAR FORM (cusp form of weight 12)
- REPRESENTATION THEORY: $\bigoplus_{n \geq 0} H^*(\mathrm{Hilb}^n \mathbb{A}^2, \mathbb{Q})$ irreducible representation of the HEISENBERG ALGEBRA
- CURVE COUNTING ON K3 SURFACES
- NAKAJIMA VARIETIES, SYMPLECTIC GEOMETRY

...

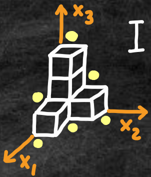
ONE CAN COMPUTE $Z_{\mathbb{A}^d}$ IN TERMS OF (HIGHER DIM) PARTITIONS

e.g. $\boxed{d=3}$: enumeration of PLANE PARTITIONS ($\leftrightarrow$ monomial ideals in 3 variables)

STRING THEORY, BPS STATES COUNTS



e.g.



$$I = (x_1^2, x_2^2, x_1x_2, x_1x_3, x_2x_3, x_3^3)$$

$n = 5$

[MacMahon function]

CHEAH, MACMAHON

$$Z_{\mathbb{A}^3}(t) = \prod_{k\geq 1} (1-t^k)^{-k}$$

... However, there is no such formula for $Z_{\mathbb{A}^d}$ if $d > 3$.

In fact, even for $d = 2, 3$ the existence of a closed formula for $p(n)$, or $p_2(n) = |\{\text{plane partitions of } n\}|$ remains a mistery.

Thank you for your attention !