Periodic and quasi-periodic vortex patch motion

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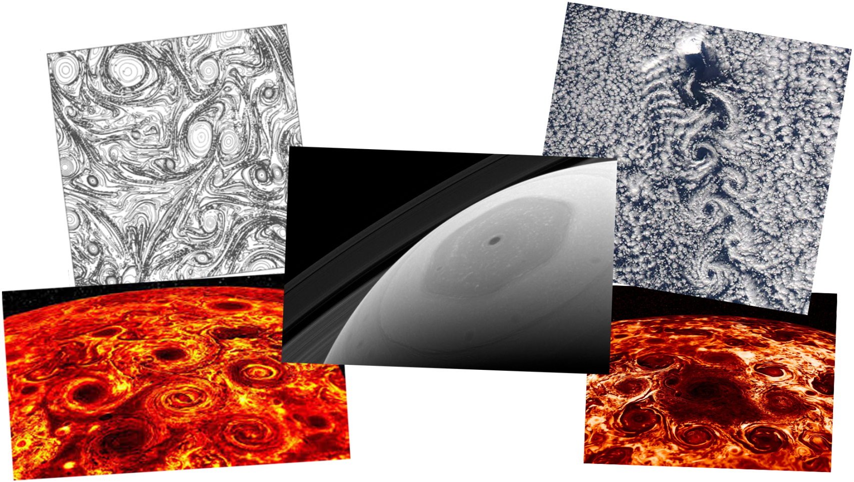

Jan Mayen Island
von Karman vortices

1. A toy model for hurricanes
2. Uniformly rotating solutions : V-states
3. Quasi-periodic vortex patches

How to model a hurricane ?

Choose a model of fluid mecanics

1. Euler / Navier-Stokes
2. Euler-α
3. Boussinesq
4. Surface Quasi-Geostrophic (SQG)
5. Quasi-Geostrophic Shallow-Water (QGSW)
6. Water-waves
7. Magneto-Hydrodynamics (MHD)

Compressible / incompressible

In $\mathbb{R}^2$, in $\mathbb{R}^3$, in bounded domain (Ω, $\mathbb{D}$), in $\mathbb{S}^2$, in $\mathbb{T}^2$, in $\mathbb{T}^3$...

Nonlinear and nonlocal transport-type fluid models

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, & \text{in } \mathbb{R}_+ \times D \\ v = \nabla^\perp \psi \\ \omega(0,\cdot) = \omega_0 \end{cases} \qquad \nabla^\perp = \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} \end{pmatrix}$$

1. Euler equations in the plane : $D = \mathbb{R}^2$

$$\psi_{\mathbf{E}}(t,z) = \Delta^{-1}\omega(t,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|z-\xi|)\omega(t,\xi)d\xi$$

2. Euler equations in the unit disc $\mathbb{D}$: $D = \mathbb{D}$

$$\psi_{\mathbb{D}}(t,z) = \Delta^{-1}\omega(t,z) = \frac{1}{4\pi}\int_{\mathbb{D}} \log\left(\left|\frac{z-\xi}{1-z\overline{\xi}}\right|^2\right)\omega(t,\xi)\,d\xi$$

3. Quasi-geostrophic shallow-water equations : $D = \mathbb{R}^2$

$$\psi_{\mathbf{SW}}^{\lambda}(t,z) = (\Delta - \lambda^2)^{-1}\omega(t,z) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} K_0\left(\lambda\,|z-\xi|\right)\omega(t,\xi)\,d\xi$$

K_0 modified Bessel function of second kind

$$K_0(z) = \log(z)F(z) + G(z), \qquad F, G \quad \text{analytic}, \qquad \lambda = \frac{\omega_c}{\sqrt{gH}} > 0$$

4. $D = \mathbb{R}^2$: $(SQG)_\alpha$ $\psi_{\mathbf{SQG}} = -(-\Delta)^{-1+\alpha/2}\omega$, Euler-$\alpha$ $\psi_\alpha = \psi_{\mathbf{E}} - \psi_{\mathbf{SW}}^{1/\alpha}$

Yudovitch-type solutions : vortex patches

Votex patch :

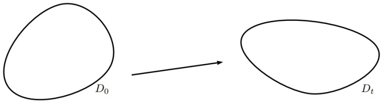
$$\omega(0,\cdot) = 1_{D_0} \in L^\infty(D) \cap L^1(D)$$

with $D_0 \subset\subset D$ (smooth) bounded domain.

$$\partial_t \omega + v \cdot \nabla \omega = 0 \quad \Rightarrow \quad \omega(t,\cdot) = 1_{D_t}$$

where

$$D_t = \Phi_t(D_0), \qquad \Phi_t(z) = z + \int_0^t v(s,\Phi_s(z))ds$$



Vortex patch equation :

$$\big[\partial_t z(t,s) - v(t,z(t,s))\big] \cdot n(t,z(t,s)) = 0$$

where $z(t,\cdot)$ is a parametrization of ∂D_t.

Stationary solutions and problem

Any radial profile patch is a stationary solution called *trivial equilibrium state*. In particular :

1. the Rankine vortices given by the discs (1858)
2. the annuli

$$A_b = \{z \in \mathbb{C} \quad \text{s.t.} \quad b < |z| < 1\}, \qquad b \in (0,1)$$

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Question : Can we find periodic or quasi-periodic solutions close to these stationary solutions ? ? ?

V-states

Definition

V-states are solutions in the form

$$\omega(t,\cdot) = 1_{D_t}, \qquad D_t = e^{i\Omega t} D_0$$

They are periodic with frequency $\Omega \in \mathbb{R}$

Stationary formulation

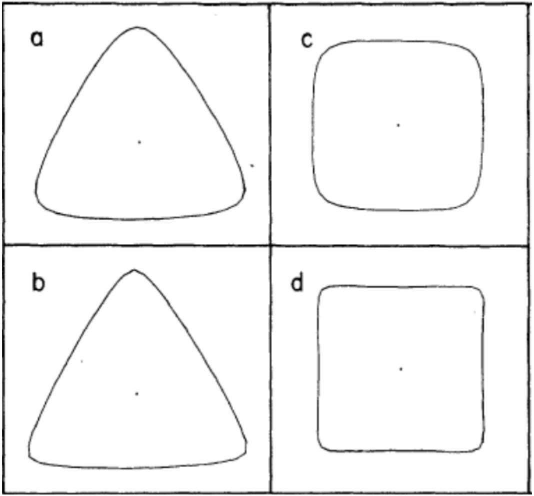
$$\left(v(0,x) - \Omega x^{\perp}\right) \cdot n(x) = 0$$

Kirchhoff (1876) : ellipses for Euler equations in the plane

$$\Omega = \frac{ab}{(a+b)^2}$$

Numerical simulation of "V-states" for Euler equations

[Deem-Zabusky '78] m-fold "V-states" ($m \in \{3, 4\}$)



How to find them analytically ?

Crandall-Rabinowitz theorem : bifurcation theory

X and Y Banach. $V \in \mathcal{V}_X(0)$

$$\begin{array}{rccc} F: & \mathbb{R} \times V & \to & Y \\ & (\Omega, x) & \mapsto & F(\Omega, x) \end{array}$$

of class C^1 such that :

1. $\forall \Omega \in \mathbb{R}, F(\Omega, 0) = 0,$
2. $\ker(D_x F(0,0)) = \langle x_0 \rangle$ et $\mathrm{Im}(D_x F(0,0))$ closed with codimension 1,
3. $\partial_\Omega D_x F(0,0) x_0 \notin \mathrm{Im}(D_x F(0,0))$ (transversality condition).

If $X = \ker(D_x F(0,0)) \oplus \mathcal{X}$, then $\exists\ U \in \mathcal{V}_{\mathbb{R}\times X}(0,0)$ in $\mathbb{R} \times X$, $a > 0$ and two continuous functions $\psi :]-a, a[\to \mathbb{R}$ and $\phi :]-a, a[\to \mathcal{X}$ st $\psi(0) = 0$, $\phi(0) = 0$ and

$$\{(\Omega, x) \in U / F(\Omega, x) = 0\} = \{(\psi(s), s x_0 + s\phi(s)) / |s| < a\} \cup \{(\Omega, 0) \in U\}.$$

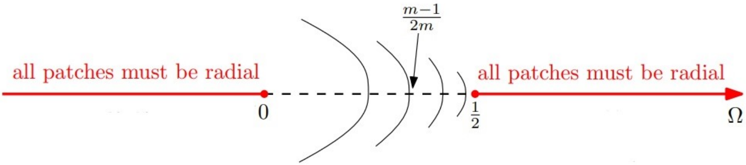
Simply-connected V-states for Euler equations in $\mathbb{R}^2$

Theorem [Burbea '82]

For all $m \geqslant 2$, there exists a branch of m-fold simply-connected V-states bifurcating from the unit disc $\mathbb{D}$ at the angular velocity

$$\Omega_m = \frac{m-1}{2m}$$

for Euler equations.



Ideas of the proof (application of Crandall-Rabinowitz's Theorem)

Conformal mappings $\Phi = \mathrm{Id} + f$: reformulation of VP eq

$$F(\Omega, f) = 0 \qquad \text{with} \qquad F(\Omega, 0) = 0.$$

1. Study of the C^1 regularity of $F : \mathbb{R} \times C^{1+\alpha}(\mathbb{T}) \to C^{\alpha}(\mathbb{T})$, $\alpha \in (0,1)$
2. Linearized operator

$$d_f F(\Omega, 0)\left[\sum_{n=0}^{\infty} a_n \overline{w}^n\right] = \sum_{n=0}^{\infty}(n+1)(\Omega_{n+1} - \Omega)a_n \mathrm{Im}(w^{n+1}), \qquad w \in \mathbb{T}$$

3. The sequence $(\Omega_n)_{n\in\mathbb{N}^*}$ is strictly monotonic

$$\ker\big(d_f F(\Omega, 0)\big) = \langle w \mapsto \overline{w}^{\mathbf{m}-1}\rangle \text{ (up to } \mathbf{m}\text{-fold symmetry restriction)}$$

4. Range close of codimension one (Fredholmness argument)

$$R\big(d_f F(\Omega, 0)\big) = \langle w \mapsto \mathrm{Im}(w^{\mathbf{m}})\rangle^{\perp}$$

5. Transversality

$$d_f F(\Omega, 0)[w \mapsto \overline{w}^{\mathbf{m}-1}] = -\mathbf{m}\mathrm{Im}(w^{\mathbf{m}})$$

Doubly-connected V-states for Euler equations in $\mathbb{R}^2$

Theorem [Hmidi-de la Hoz-Mateu-Verdera '14]

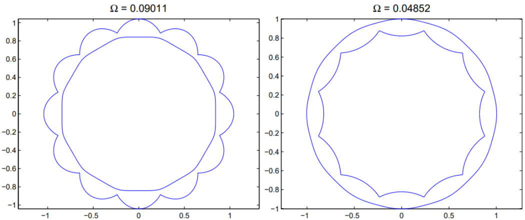
Let $b \in (0,1)$ For any $m \in \mathbb{N}^*$ st

$$1 + b^m - \frac{1-b^2}{2} m < 0$$

there exist two branches of m-fold doubly-connected V-states bifurcating from the annulus A_b at the angular velocities

$$\Omega_m^{\pm}(b) = \frac{1-b^2}{4} \pm \frac{1}{2m}\sqrt{\left(\frac{m(1-b^2)}{2} - 1\right)^2 - b^{2m}}$$

for Euler equations.



Additional references

For the simply-connected case

1. Regularity : [Hmidi-Mateu-Verdera '12], [Castro-Córdoba-Gómez-Serrano '15, '18]
2. Global bifurcation diagram : [Fraenkel '00], [Hmidi '14], [Gomez-Serrano-Park-Shi-Yao '20], [Hassainia-Masmoudi-Wheeler '17],
3. Bifurcation from the ellipse : [Hmidi-Mateu '15]
4. V-states $(SQG)_\alpha$: [Hassainia-Hmidi '14]

For the doubly-connected case

1. Necessary confocal ellipses : [Hmidi-Mateu-Verdera '13]
2. Global bifurcation diagram : [Gomez-Serrano-Park-Shi-Yao '20]
3. Degenerate bifurcation : [Hmidi-Mateu '15]
4. V-states $(SQG)_\alpha$: [Hassainia-Hmidi-Hoz '15], [Renault '17]
5. Stationary $(SQG)_\alpha$: [Gómez-Serrano '18]

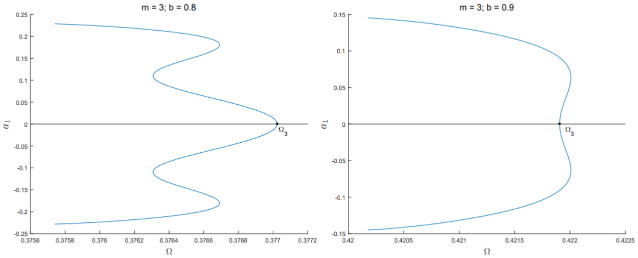
Simply-connected V-states for Euler equations in $\mathbb{D}$

Theorem [Hassainia-Hmidi-de la Hoz-Mateu '15]

Let $b \in (0,1)$. For all $\mathrm{m} \geqslant 1$ there exists a branch of m-fold simply-connected V-states bifurcating from the disc $b\mathbb{D}$ at the angular velocity

$$\Omega_{\mathrm{m}}(b) = \frac{\mathrm{m} - 1 + b^{2\mathrm{m}}}{2\mathrm{m}}$$

for Euler equations in the unit disc.



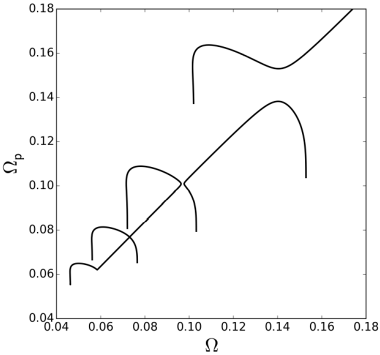
Simply-connected V-states for QGSW

Theorem [Dritschel-Hmidi-Renault '18]

Let $\lambda \in \mathbb{R}$. For all $m \geqslant 1$ there exists a branch of m-fold simply-connected V-states bifurcating from the unit disc $\mathbb{D}$ at the angular velocity

$$\Omega_m(\lambda) = I_1(\lambda)K_1(\lambda) - I_m(\lambda)K_m(\lambda)$$

for $(QGSW)_\lambda$ equations.



Doubly-connected V-states for QGSW

Theorem [R. '22]

Let $(\lambda, b) \in (0,\infty) \times (0,1)$. There exists $N(\lambda, b) \in \mathbb{N}^*$ st for any $\mathrm{m} \in \mathbb{N}^*$ with $\mathrm{m} \geqslant N(\lambda, b)$ there exist two branches of m-fold doubly-connected V-states bifurcating from the annulus A_b at the angular velocities

$$\Omega_{\mathrm{m}}^{\pm}(\lambda, b) = \frac{1-b^2}{2b}\Lambda_1(\lambda, b) + \frac{1}{2}\left(\Omega_{\mathrm{m}}(\lambda) - \Omega_{\mathrm{m}}(\lambda b)\right)$$
$$\pm \frac{1}{2b}\sqrt{\left[b(\Omega_{\mathrm{m}}(\lambda) + \Omega_{\mathrm{m}}(\lambda b)) - (1+b^2)\Lambda_1(\lambda, b)\right]^2 - 4b^2\Lambda_{\mathrm{m}}^2(\lambda, b)}$$

for $(QGSW)_\lambda$, where

$$\Lambda_n(\lambda, b) = I_n(\lambda b)K_n(\lambda)$$

and

$$\Omega_n(x) = I_1(x)K_1(x) - I_n(x)K_n(x)$$

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for $(QGSW)_\lambda$, where

$$\Lambda_n(\lambda, b) = I_n(\lambda b)K_n(\lambda)$$

and

$$\Omega_n(x) = I_1(x)K_1(x) - I_n(x)K_n(x)$$

Remark : Continuity of the spectrum (doubly-connected Euler and simply-connected QGSW)

Ideas of the proof (application of Crandall-Rabinowitz's Theorem)

Conformal mappings $\Phi_1 = \mathrm{Id} + f_1$ and $\Phi_2 = b\mathrm{Id} + f_2$: reformulation

$$G(\lambda, b, \Omega, f_1, f_2) = 0 \qquad \text{with} \qquad G(\lambda, b, \Omega, 0, 0) = 0.$$

1. Study of the C^1 regularity of G
2. Linearized operator

$$d_{(f_1,f_2)}G(\lambda, b, \Omega, 0, 0)\left[\sum_{n=0}^{\infty} a_n \overline{w}^n, \sum_{n=0}^{\infty} b_n \overline{w}^n\right] = \sum_{n=0}^{\infty} M_{n+1}(\lambda, b, \Omega)\begin{pmatrix} a_n \\ b_n \end{pmatrix} \mathrm{Im}(w^{n+1})$$

3. $M_n(\lambda, b, \Omega)$ singular iff $\Omega = \Omega_n^{\pm}(\lambda, b)$ for $n \geqslant N(\lambda, b)$
4. The sequences $(\Omega_n^{\pm}(\lambda, b))_{n \geqslant N(\lambda, b)}$ are strictly monotonic (asymp. Bessel) $\rightsquigarrow$ one dimensional kernel (up to symmetry restriction)
5. Explicit range (closed+codimension one)
6. Transversality : simple eigenvalues + definition of $\Omega_n^{\pm}(\lambda, b)$
7. Analyticity of the boundaries : elliptic free boundary problem

Simply-connected V-states for Euler-α

Theorem [R. '22]

Let $\alpha > 0$. For all $m \geqslant 1$ there exists a branch of m-fold simply-connected V-states bifurcating from the unit disc $\mathbb{D}$ at the angular velocity

$$\Omega_m(\alpha) = \frac{m-1}{2m} - \left(I_1\left(\tfrac{1}{\alpha}\right)K_1\left(\tfrac{1}{\alpha}\right) - I_m\left(\tfrac{1}{\alpha}\right)K_m\left(\tfrac{1}{\alpha}\right)\right)$$

for Euler-α equations.

Doubly-connected ?

Simply-connected V-states for Euler-α

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Let $\alpha > 0$. For all $m \geqslant 1$ there exists a branch of m-fold simply-connected V-states bifurcating from the unit disc $\mathbb{D}$ at the angular velocity

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for Euler-α equations.

Doubly-connected ? GIVE ME MORE TIME ! !

Time quasi-periodic functions

Definition

A function $f:\mathbb{R}\to\mathbb{C}$ is quasi-periodic iff

$$f(t)=\widetilde{f}(\omega t),\qquad \widetilde{f}=\widetilde{f}(\varphi):\mathbb{T}^d\to\mathbb{C}$$

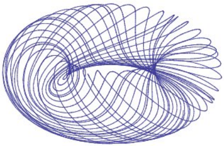
with

$$\omega=(\omega_1,\ldots,\omega_d)\in\mathbb{R}^d,\qquad \forall l\in\mathbb{Z}^d\backslash\{0\},\quad \omega\cdot l\neq 0$$

Fundamental example :

$$f(t)=\sum_{j=1}^{d} f_j e^{i\omega_j t},\qquad f_j\in\mathbb{C}$$

Trajectory contained in a torus of dimension d with frequency ω : dense orbits.



History

Integrable Hamiltonian (Liouville)

$$\begin{cases} \dot p = -\nabla_q H(p,q) \\ \dot q = \nabla_p H(p,q) \end{cases} \overset{(p,q)=A(I,\vartheta)}{\rightsquigarrow} \begin{cases} \dot I = 0 \\ \dot\vartheta = \nabla_I h(I) = \omega(I) \end{cases}$$

Hence

$$I(t) = I_0 \in \mathbb{R}^d, \qquad \vartheta(t) = \omega(I_0)t + \vartheta_0 \in \mathbb{T}^d$$

Persistence under perturbation : [Kolmogorov '54], [Arnold '63], [Moser '62] (KAM)

$$H(I,\vartheta) = h(I) + \varepsilon P(I,\vartheta)$$

Tool : Newton method $\rightsquigarrow$ Almost all invariant tori survive

Small divisors problem

Solve for a given g (with zero average)

$$\omega \cdot \partial_\varphi f = g$$

Fourier

$$f(\varphi) = \sum_{l \in \mathbb{Z}^d \setminus \{0\}} \frac{g_l}{i\omega \cdot l} e^{il\cdot\varphi}$$

Problem : Diophantine approximations at the denominator

$$\mathtt{DC}(\gamma, \tau) = \bigcap_{l \in \mathbb{Z}^d \setminus \{0\}} \left\{ \omega \in \mathbb{R}^d \quad \text{t.q.} \quad |\omega \cdot l| > \frac{\gamma}{\langle l \rangle^{\tau}} \right\}$$

Estimate with fixed loss of regularity

$$\|f\|_{H^s} \lesssim \gamma^{-1} \|g\|_{H^{s+\tau}}$$

Nash-Moser scheme [Nash '54]

$F(u)=0 \rightsquigarrow$ Newton scheme

1. $\|F(u_0)\|_{s_0+\tau} \lesssim \varepsilon$ and F, dF, d^2F with tame estimates

$$\|F(u)\|_s \lesssim 1+\|u\|_{s+\alpha}, \qquad \|dF(u)[h]\|_s \lesssim \|u\|_{s_0}\|h\|_{s+\alpha}+\|u\|_{s+\alpha}\|h\|_{s_0}$$

2. Right inverse with loss of derivative

$$dF(u)\circ T(u)=\mathrm{Id}, \qquad \|T(u)[h]\|_s \lesssim \|u\|_{s_0}\|h\|_{s+\tau}+\|u\|_{s+\tau}\|h\|_{s_0}$$

Π_N projector in Fourier up to modes of size N

$$u_{n+1}=u_n-\Pi_{N_n}T(u_n)F(u_n), \qquad N_n=N_0^{(3/2)^n}$$

$$\|u_{n+1}-u_n\|_{s_0} \lesssim N_n^{\alpha+\tau-\beta}\|T(u_{n-1})F(u_{n-1})\|_{s_0+\beta}+N_n^{\alpha+\tau}\|u_n-u_{n-1}\|_{s_0}^2$$
$$\|F(u_n)\|_{s_0-\alpha} \lesssim N_n^{-\beta}\|T(u_{n-1})F(u_{n-1})\|_{s_0+\beta}+\|u_n-u_{n-1}\|_{s_0}^2$$

Induction

$$\|T(u_{n-1})F(u_{n-1})\|_{s_0+\beta} \lesssim N_n^{\nu}, \qquad \|u_n-u_{n-1}\|_{s_0} \lesssim N_n^{-\nu}$$

Play with 3 Sobolev indices of regularity / a priori free parameters (β,ν)

KAM for PDE

1. NLS/NLW : Bambusi, Berti, Bolle, Bourgain, Craig, Feola, Grébert, Kappeler, Kuksin, Procesi, Wayne, etc.. '87-now
2. [Berti-Bolle "A Nash-Moser approach to KAM theory" '15]
3. (m)KdV/Airy : Baldi, Berti, Kappeler, Montalto '12-now
4. Water-waves : Baldi, Berti, Feola, Franzoi, Giuliani, Haus, Maspero, Montalto, etc.. '16-now
5. Euler eq : [Crouseilles-Faou '12, no KAM], [Baldi-Montalto '20]

Hamiltonian equation for the radial deformation

Polar parametrization of ∂D_t :

$$z(t,\theta) = R(t,\theta)e^{i\theta}, \qquad R(t,\theta) = \sqrt{1+2r(t,\theta)}$$

and small radial deformation

$$|r(t,\theta)| \ll 1$$

vp eq $\rightsquigarrow$ nonlinear and nonlocal Hamiltonian transport PDE

$$\partial_t r = \partial_\theta \nabla_{L^2_\theta(\mathbb{T})} H(r)$$

where H is the kinetic energy

$$H(r)(t) = -\frac{1}{4\pi} \int_{D_t} \psi(t,z) dz$$

1. $r = 0$ stationary solution (disc)
2. Conserved space average :

$$\partial_t \int_{\mathbb{T}} r(t,\theta)d\theta = 0$$

Phase space

$$L_0^2(\mathbb{T},\mathbb{R}) = \left\{ r(\theta) = \sum_{j\in\mathbb{Z}^*} r_j e^{ij\theta} \quad \text{s.t.} \quad r_{-j} = \overline{r_j}, \sum_{j\in\mathbb{Z}^*} |r_j|^2 < +\infty \right\}$$

3. Reversibility $(t,\theta) \mapsto r(t,\theta)$ sol $\Rightarrow (t,\theta) \mapsto r(-t,-\theta)$ sol

$$H \circ \mathscr{S} = H, \qquad \mathscr{S}(r)(\theta) = r(-\theta)$$

Look for solutions in the form

$$r(t,\theta) = \sum_{j\in\mathbb{N}^*} a_j(\omega t)\cos(j\theta) + b_j(\omega t)\sin(j\theta), \qquad a_j \text{ even and } b_j \text{ odd}$$

Linearized operator at a general state r close to 0 for QGSW eq

$$\mathcal{L}_r = \underbrace{\partial_t + \partial_\theta \left(V_r(\lambda) \cdot \right)}_{\text{transport}} - \underbrace{\partial_\theta \mathrm{L}_{r,\lambda}}_{\in OPS^0}$$

where

$$V_r(\lambda, t, \theta) = \int_{\mathbb{T}} K_0 \left(\lambda A_r(t, \theta, \eta) \right) \partial_\eta \left(\frac{R(t,\eta) \sin(\eta - \theta)}{R(t,\theta)} \right) d\eta$$

and

$$\mathrm{L}_{r,\lambda} \rho(t,\theta) = \int_{\mathbb{T}} K_0 \left(\lambda A_r(t,\theta,\eta) \right) \rho(t,\eta) d\eta$$

with

$$A_r(t,\theta,\eta) = \left| R(t,\theta) e^{i\theta} - R(t,\eta) e^{i\eta} \right| = 2 \left| \sin\left(\frac{\eta - \theta}{2} \right) \right| (1 + O(r))$$

Linearized operator at the equilibrium state $r = 0$

$$\mathcal{L}_0 = \partial_t + V_0(\lambda)\partial_\theta + \partial_\theta \mathcal{K}_\lambda * \cdot$$

with

$$V_0(\lambda) = I_1(\lambda)K_1(\lambda), \quad \mathcal{K}_\lambda(\theta) = -K_0(\lambda|\sin(\tfrac{\theta}{2})|)$$

We have

$$\mathcal{L}_0\rho = 0 \Leftrightarrow \dot{\rho}_j = -i\Omega_j(\lambda)\rho_j \Leftrightarrow \rho_j(t) = \rho_j(0)e^{-i\Omega_j(\lambda)t}$$

with

$$\Omega_j(\lambda) = j(I_1(\lambda)K_1(\lambda) - I_{|j|}(\lambda)K_{|j|}(\lambda))$$

Reversible solutions :

$$(t,\theta) \mapsto \sum_{j\in\mathbb{N}^*} \rho_j(0)\cos(j\theta - \Omega_j(\lambda)t)$$

Quasi-periodic solutions at the linear level

Let $\mathbb{S} = \{j_1, \ldots, j_d\} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. Set $\omega_{\mathbf{Eq}}(\lambda) = (\Omega_j(\lambda))_{j\in\mathbb{S}}$.
Quasi-periodic reversible solutions :

$$(t,\theta) \mapsto \sum_{j\in\mathbb{S}} r_j \cos(j\theta - \Omega_j(\lambda)t), \quad r_j \in \mathbb{R}$$

$$\mathcal{C}^{\gamma}_{\mathbf{Eq}} = \bigcap_{l\in\mathbb{Z}^d\setminus\{0\}} \left\{ \lambda \in [\lambda_0, \lambda_1] \quad \text{s.t.} \quad |\omega_{\mathbf{Eq}}(\lambda)\cdot l| > \frac{\gamma}{\langle l\rangle^{\tau}} \right\}$$

How to measure ?

Quasi-periodic solutions at the linear level

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Quasi-periodic reversible solutions :

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How to measure ?

Rüssmann's Lemma

Let $q_0 \in \mathbb{N}^*$, $(a,b) \in \mathbb{R}^2$, $(\mathtt{m}, \mathtt{M}) \in (\mathbb{R}_+^*)^2$, $f \in C^{q_0}([a,b],\mathbb{R})$ st

$$\inf_{x\in[a,b]} \max_{k\in[\![0,q_0]\!]} |f^{(k)}(x)| \geqslant \mathtt{m} \qquad \text{"transversality condition"}$$

then

$$\left|\{x \in [a,b] \quad \text{s.t.} \quad |f(x)| \leqslant \mathtt{M}\}\right| \lesssim \frac{\mathtt{M}^{\frac{1}{q_0}}}{\mathtt{m}^{1+\frac{1}{q_0}}}$$

Rüssmann conditions for the equilibrium (transversality)

There exist q_0 and $\rho_0 > 0$ such that

$$\forall l \in \mathbb{Z}^d \backslash \{0\}, \quad \inf_{\lambda \in [\lambda_0, \lambda_1]} \max_{q \in [\![0, q_0]\!]} |\partial_\lambda^q \omega_{\mathbf{Eq}}(\lambda) \cdot l| \geqslant \rho_0 \langle l \rangle$$

1. Proof by contradiction.
2. Non-degeneracy : $\lambda \mapsto \omega_{\mathbf{Eq}}(\lambda)$ is analytic and not contained in an hyperplane.
3. Asymptotics and integral representations of modified Bessel functions.

Theorem [Hmidi-R. '21]

Let $[\lambda_0,\lambda_1]\subset(0,\infty)$ and $\mathbb{S}\subset\mathbb{N}^*$ finite.
There exists $\varepsilon\in(0,1)$ st for $(\varepsilon_j)_{j\in\mathbb{S}}\in(\mathbb{R}^*)^{|\mathbb{S}|}$ with $|\varepsilon_j|\leqslant\varepsilon$ there exists a Cantor set $\mathcal{C}\subset(\lambda_0,\lambda_1)$

$$\lim_{\varepsilon\to0}|\mathcal{C}|=\lambda_1-\lambda_0$$

st for all $\lambda\in\mathcal{C}$, the $(QGSW)_\lambda$ equations admit a time quasi-periodic vortex patch solution with frequency vector $\omega(\lambda,\varepsilon)=(\Omega_j(\lambda,\varepsilon))_{j\in\mathbb{S}}$ close to $\omega_{\mathrm{Eq}}(\lambda)=(\Omega_j(\lambda))_{j\in\mathbb{S}}$ in the form

$$\mathbf{1}_{D_t},\qquad D_t=\left\{re^{i\theta},\,\theta\in[0,2\pi],\,0\leqslant r\leqslant\sqrt{1+2r(t,\theta)}\right\}$$

$$r(t,\theta)=\sum_{j\in\mathbb{S}}\varepsilon_j\cos\left(\Omega_j(\lambda,\varepsilon)t-j\theta\right)+\mathtt{p}(\omega(\lambda,\varepsilon)t,\theta)$$

In addition, for $s=s(|\mathbb{S}|,q_0)$ large enough $\|\mathtt{p}\|_{H^s(\mathbb{T}^{|\mathbb{S}|+1},\mathbb{R})}\underset{\varepsilon\to0}{=}o(\varepsilon)$.

$$\Omega_j(\lambda,\varepsilon)=\Omega_j(\lambda)+O(\varepsilon)$$

Theorem [Hmidi-R. '21]

Let $[\lambda_0, \lambda_1] \subset (0, \infty)$ and $\mathbb{S} \subset \mathbb{N}^*$ finite.
There exists $\varepsilon \in (0,1)$ st for $(\varepsilon_j)_{j\in\mathbb{S}} \in (\mathbb{R}^*)^{|\mathbb{S}|}$ with $|\varepsilon_j| \leqslant \varepsilon$ there exists a Cantor set $\mathcal{C} \subset (\lambda_0, \lambda_1)$

$$\lim_{\varepsilon\to 0} |\mathcal{C}| = \lambda_1 - \lambda_0$$

st for all $\lambda \in \mathcal{C}$, the $(QGSW)_\lambda$ equations admit a time quasi-periodic vortex patch solution with frequency vector $\omega(\lambda, \varepsilon) = (\Omega_j(\lambda, \varepsilon))_{j\in\mathbb{S}}$ close to $\omega_{\mathbf{Eq}}(\lambda) = (\Omega_j(\lambda))_{j\in\mathbb{S}}$ in the form

$$\mathbf{1}_{D_t}, \qquad D_t = \left\{ re^{i\theta},\ \theta \in [0, 2\pi],\ 0 \leqslant r \leqslant \sqrt{1 + 2r(t,\theta)} \right\}$$

$$r(t,\theta) = \sum_{j\in\mathbb{S}} \varepsilon_j \cos\left(\Omega_j(\lambda,\varepsilon)t - j\theta\right) + \mathrm{p}(\omega(\lambda,\varepsilon)t, \theta)$$

In addition, for $s = s(|\mathbb{S}|, q_0)$ large enough $\|\mathrm{p}\|_{H^s(\mathbb{T}^{|\mathbb{S}|+1},\mathbb{R})} \underset{\varepsilon\to 0}{=} o(\varepsilon)$.

$$\Omega_j(\lambda, \varepsilon) = \Omega_j(\lambda) + O(\varepsilon)$$

Remark : The restriction to $\mathcal{C}$ is not technical.

1. Rescalling $r \rightsquigarrow \varepsilon r$, quasilinear perturbation of the linearized eq at equilibrium
$$\partial_\theta \nabla H(r) = \partial_\theta \nabla^2 H(0)(r) + \varepsilon P_\varepsilon(r) \rightsquigarrow KAM$$
2. $L^2_0 = L_{\mathbb{S}} \oplus L^2_\perp$ action-angle-normal variables $i = (\vartheta, I, z) \rightsquigarrow$ reformulation with embedded tori $\mathcal{F}_\varepsilon(i, \lambda, \omega) = 0$
3. Nash-Moser scheme : find an approximate right inverse with tame estimates of $d_i\mathcal{F}(i)$
Berti-Bolle theory : Conjugate to a triangular system + error terms.
To solve the triangular system : inversion on the normal directions.
4. Quasilinear effects on the linearized operator : conjugation to constant coeff (KAM-type reductions, symplectic quasi-periodic CVAR, Toeplitz topology, Cantor sets, error terms)
5. Rigidification of ω, measure of the final Cantor set (perturbed Russmann-transversality conditions)

Results on simply-connected quasi-periodic vortex patches

Key idea : Play with a parameter to capt these solutions close to equilibrium states up to selecting the parameter in a Cantor set.

1. $(SQG)_\alpha$: [Hassainia-Hmidi-Masmoudi '21] close to $\mathbb{D}$
2. $(QGSW)_\lambda$: [Hmidi-R. '21] close to $\mathbb{D}$
3. Euler eq : [Berti-Hassainia-Masmoudi '22] close to the ellipse
   Parameter linked to the excentricity
4. Euler eq in $\mathbb{D}$: [Hassainia-R. '22] close to $b\mathbb{D}$ for $b \in (0,1)$
5. Euler-α in $\mathbb{R}^2$: [R. '22] close to $\mathbb{D}$

Quasi-periodic in time doubly-connected vortex patches

Theorem (Part I) [Hassainia-Hmidi-R. 22']

Let $b^* \in (0,1)$ (except some values). There exists $\mathtt{m}^* \geqslant 3$ st for any

$$\mathtt{m} \geqslant \mathtt{m}^*, \qquad \mathbb{S}_1, \mathbb{S}_2 \subset \mathtt{m}\mathbb{N}^* \text{ finite}, \qquad \mathbb{S}_1 \cap \mathbb{S}_2 = \varnothing$$

there exists $\varepsilon \in (0,1)$ st for any $(\varepsilon_{j,k})_{j \in \mathbb{S}_k}$ with $|\varepsilon_{j,k}| \leqslant \varepsilon$ $k \in \{1,2\}$, there exists a Cantor set $\mathcal{C} \subset (0, b^*)$ with

$$\lim_{\varepsilon \to 0} |\mathcal{C}| = b^*$$

st for all $b \in \mathcal{C}$, the Euler equations admit a time quasi-periodic doubly-connected vortex patch solution with frequency vector

$$\omega(b,\varepsilon) = \left((\Omega_{j,1}(b,\varepsilon))_{j \in \mathbb{S}_1}, (\Omega_{j,2}(b,\varepsilon))_{j \in \mathbb{S}_2}\right)$$

close to

$$\omega_{\mathrm{Eq}}(b) = \left((j\Omega_j^+(b))_{j \in \mathbb{S}_1}, (j\Omega_j^-(b))_{j \in \mathbb{S}_2}\right)$$

in the form

Quasi-periodic in time doubly-connected vortex patches

Theorem (Part II) [Hassainia-Hmidi-R. 22']

$$\mathbf{1}_{D_t}, \qquad D_t = \left\{re^{i\theta},\ \theta \in [0,2\pi],\ \sqrt{b^2+2r_2(t,\theta)} \leqslant r \leqslant \sqrt{1+2r_1(t,\theta)}\right\}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}(t,\theta) = \sum_{j\in\mathbb{S}_1} \varepsilon_{j,1} \cos\left(\Omega_{j,1}(b,\varepsilon)t - j\theta\right)\begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix}$$

$$+ \sum_{j\in\mathbb{S}_2} \varepsilon_{j,2} \cos\left(\Omega_{j,2}(a,\varepsilon)t - j\theta\right)\begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix} + \mathrm{p}(\omega(b,\varepsilon)t, x)$$

with

$$a_j(b) = \frac{b^j}{\frac{1-b^2}{2}j - 1 + \sqrt{\left(\frac{1-b^2}{2}j-1\right)^2 - \frac{b^{2j}}{2}}}$$

The perturbation satisfies for s large enough

$$\mathrm{p}(-\varphi,-\theta) = \mathrm{p}(\varphi,\theta) = \mathrm{p}(\varphi, \theta + \tfrac{2\pi}{\mathrm{m}}), \qquad \|\mathrm{p}\|_{H^s} = o(\varepsilon)$$

Thank you for your attention !