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TOWARDS AN EFFECTIVE-ONE-BODY MODEL FOR EXTREME-MASS-RATIO INSPIRALS

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Based on: arXiv:2208.01049 [gr-qc], arXiv:2208.02055 [gr-qc]
In collaboration with: A. Nagar, B. Wardell, A. Pound, N. Warburton

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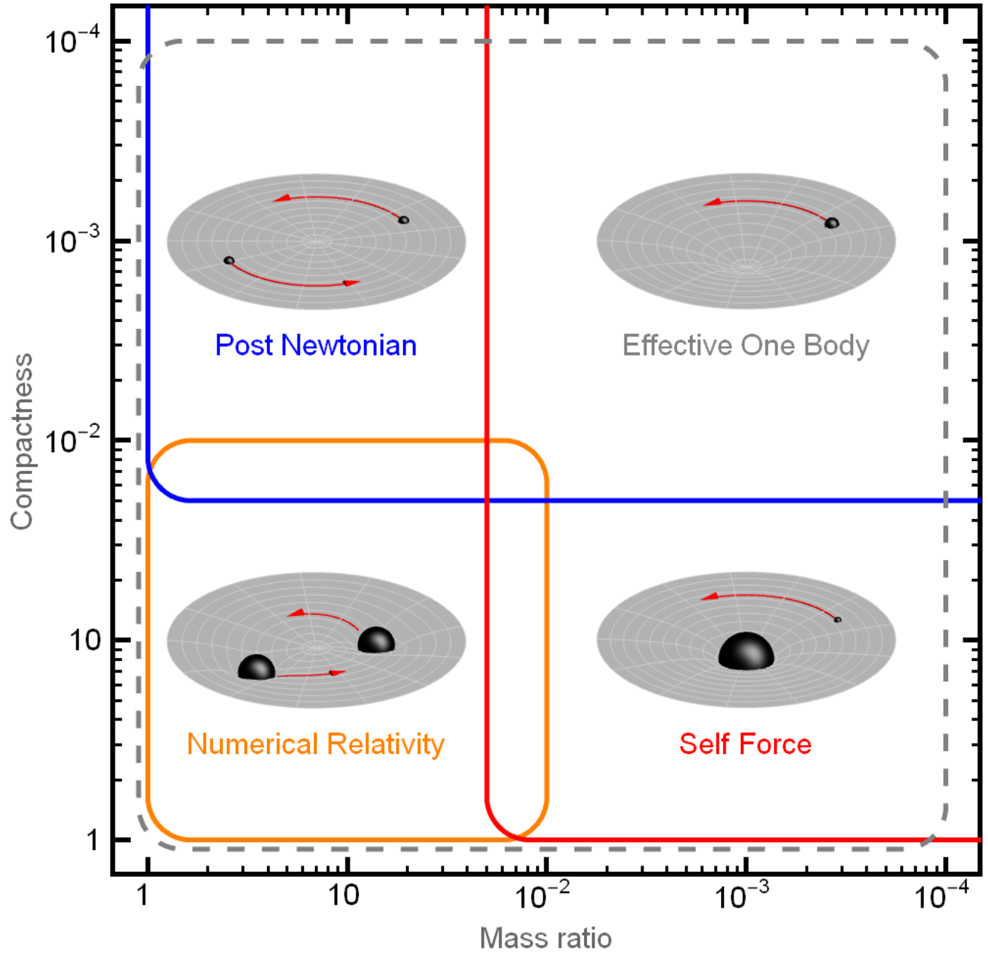
OUTLINE

- Brief intro to the effective-one-body (EOB) approach to the two-body problem in general relativity
- Comparison between the EOB model TEOBResumS and gravitational self-force (GSF) results:
  - quasi-circular equatorial motion, nonspinning black holes
  - intermediate- and extreme-mass-ratio inspirals (IMRIs and EMRIs)
- Modifying TEOBResumS for IMRIs and EMRIs
- Features to be added: spin, eccentricity,
  environment & beyond GR

THE EFFECTIVE-ONE-BODY FORMALISM



Theoretical results from classical approaches

Information from **N**umerical **R**elativity (NR)

EOB

flexible analytical approach, mapping the two-body dynamics in the motion of a particle with the reduced mass of the system moving in an effective metric (deformation of Schwarzschild/Kerr)

THEORETICAL FRAMEWORK

conservative sector

- Hamiltonian: found by mapping the "energy levels" of the real problem at a given PN order to the effective ones

dissipative sector

- Hamiltonian equations of motion complemented by the **radiation reaction**

} dynamics

- Waveform: (inspiral + plunge) + ringdown decomposed on spin-weighted spherical harmonics

DYNAMICAL BACKGROUND

$G = c = 1$

Mass ratio $q = \frac{m_1}{m_2}, \quad m_1 > m_2$ Symmetric mass ratio $\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$

Continuous deformation in ν of a Schwarzschild metric:

$$ds^2_{\text{eff}} = g^{\text{eff}}_{\mu\nu} dx^\mu_{\text{eff}} dx^\nu_{\text{eff}} = -A(r)dt^2 + B(r)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad u = 1/r$$

$$A^{\text{PN}}_{\text{orb}}(u) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \nu \left[a^c_5(\nu) + a^{\log}_5 \ln u\right] u^5 + \nu \left[a^c_6(\nu) + a^{\log}_6 \ln u\right] u^6$$

EOB Hamiltonian for nonspinning binaries:

$$\hat{H}_{\text{EOB}} \equiv \frac{H_{\text{EOB}}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_{\text{eff}} - 1\right)} \qquad \hat{H}_{\text{eff}} = \sqrt{p^2_{r_*} + A(r,\nu)\left(1 + \frac{p^2_\phi}{r^2} + 2\nu(4 - 3\nu)p^4_{r_*}\right)}$$

HAMILTONIAN EQUATIONS OF MOTION

$$\frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_\varphi} = \Omega$$

Orbital frequency

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}}$$

solved numerically with ODE solver

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi = \hat{\mathcal{F}}^\infty_\varphi + \hat{\mathcal{F}}^{\text{H}}_\varphi$$

Radiation reaction:
asymptotic contribution + horizon contribution

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial r}$$

INSPIRAL (+ PLUNGE) WAVEFORM

- Strain: $$h_+ - ih_\times = \frac{1}{D_L}\sum_\ell \sum_{m=-\ell}^{\ell} h_{\ell m}\, {}_{-2}Y_{\ell m}(\iota,\phi)$$

multipoles

Newtonian prefactor

Resumed PN correction

$$h_{\ell m} = h^{(N,\epsilon)}_{\ell m}\, \hat{S}^{\epsilon}_{\rm eff}\, \hat{h}^{\rm tail}_{\ell m}(x)\, \left[\rho_{\ell m}(x)\right]^{\ell}\, \hat{h}^{\rm NQC}_{\ell m}$$

Next-to-Quasi-Circular corrections

- Regge-Wheeler-Zerilli normalized waveform:

$$\Psi_{\ell m} \equiv \frac{h_{\ell m}}{\sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}}$$

$$\Psi_{22} = A_{22} e^{-i\phi_{22}} \qquad \omega_{22} \equiv \dot{\phi}_{22}$$

waveform frequency

TEOBRESUMS

- EOB model built for **comparable-mass binaries** (versions: quasi-circular, eccentric, precessing)
- Incorporates analytical information (PN expansions for potentials & waveform/flux, resummed in some way)
- Some parameters are tuned to NR results (orbital sector, spin-orbit, merger & ringdown)

BLACK HOLE BINARIES: HIGHER MASS RATIOS

- Intermediate and extreme mass ratio black hole binaries are among the sources of the next generation of gravitational wave detectors (ET, CE, LISA)
- Regime scarcely explored by NR
- Apart from EOB, **gravitational self-force** (GSF) theory is the only other available tool to probe the inspiral
- → comparing EOB and GSF

SECOND ORDER GRAVITATIONAL SELF-FORCE

- GSF: taking into account the deviation from the test-mass case due to the second object's gravitational field
- Expanding the metric to 2nd order in the small mass ratio: $\epsilon \equiv m_2/m_1 \leq 1$
- Two-timescale expansion: slow radiation reaction timescale vs fast orbital timescale
- We consider here the post-adiabatic (PA) model presented in Wardell et al. 2021 (arXiv:2112.12265v2)

WAVEFORM ALIGNMENT IN THE TIME DOMAIN

- We focus on the $\ell = m = 2$ strain multipole
- Phasing: finding the **time and phase shift** by minimizing the root-mean-square of the phase difference on a certain interval

warning: a bit arbitrary!

$$\Delta\phi(t_i, \tau, \alpha) = \left(\phi_2(t_i - \tau) - \alpha\right) - \phi_1(t_i)$$

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Delta\phi(t_i, \tau, \alpha))^2}$$

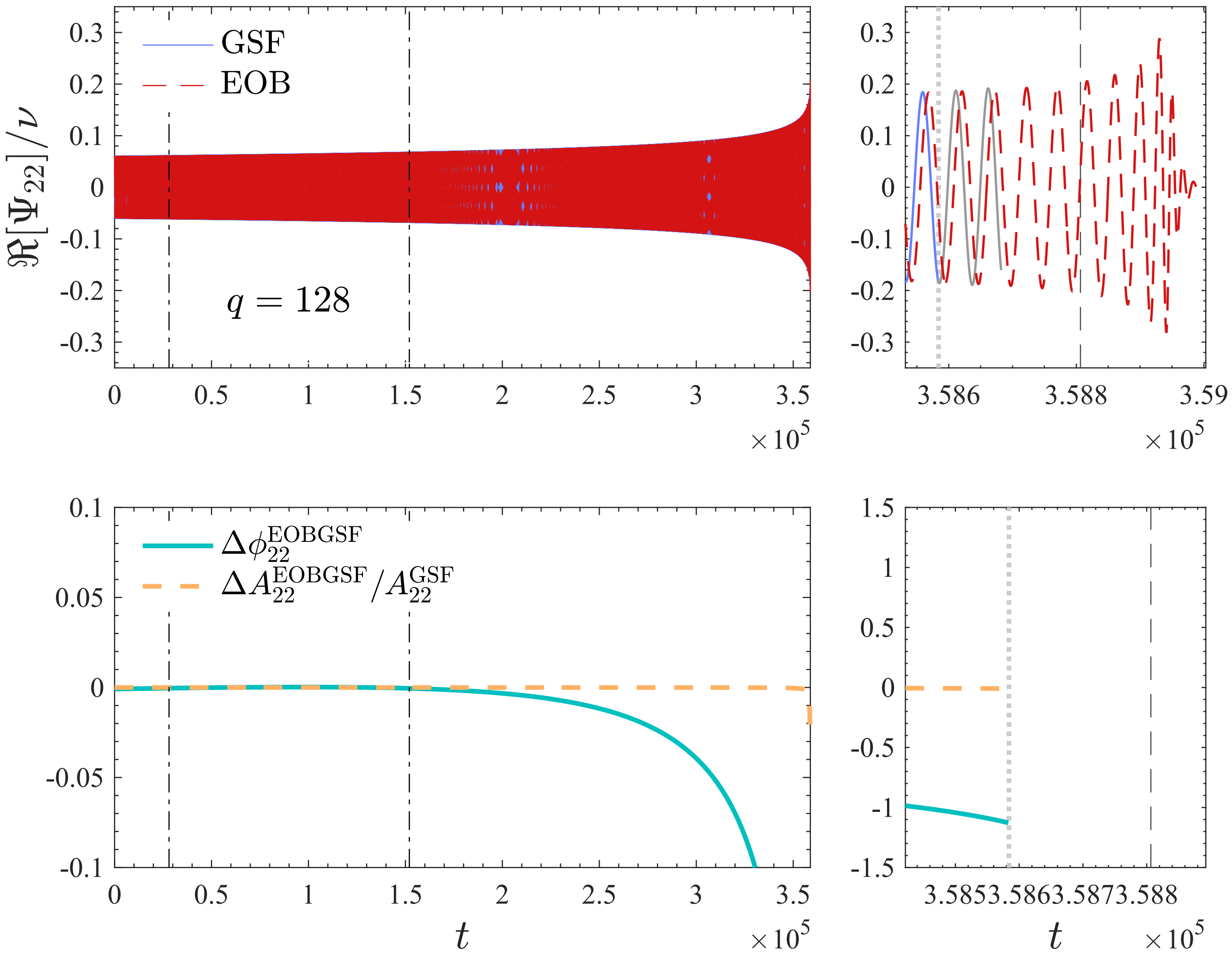
$$\Psi_{22}^{1} = A_{22}^{1}(t_1)e^{-i\phi_1(t_1)}$$

$$\Psi_{22}^{2} = A_{22}^{2}(t_2 - \tau)e^{-i[\phi_2(t_2-\tau)-\alpha]}$$

PHASE DIFFERENCES

Binaries with mass ratio q = 15, 32, 64, 128 to complement the findings of Nagar et al. 2022 (arXiv:2202.05643v1)

q	$\Delta\phi_{22,t}^{\mathrm{EOBGSF}}$
15	0.3782
32	−0.1267
64	−0.5091
128	−1.1287



GAUGE-INVARIANT ANALYSIS: Q_ω

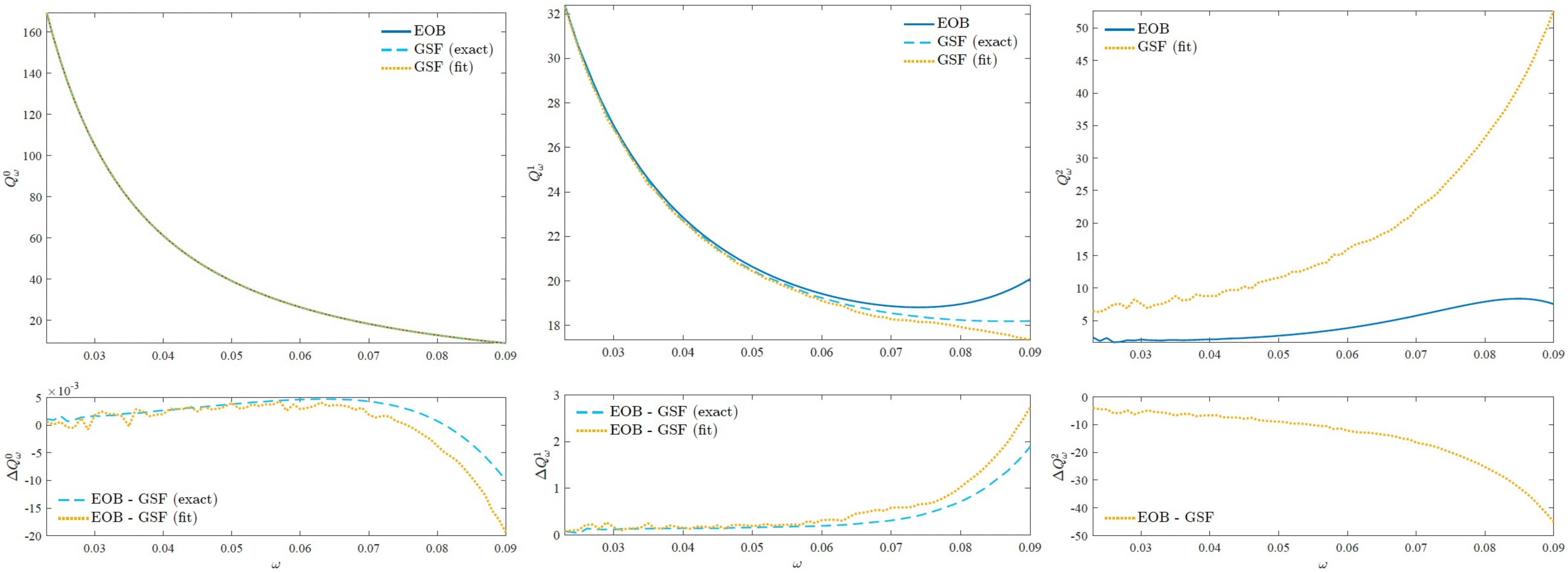
- Adiabaticity parameter: $Q_\omega = \frac{\omega^2}{\dot{\omega}}$ $\quad \omega = \omega_{22}$
- Q_ω >> 1 adiabatic motion
- Phase difference: $\Delta\phi_{(\omega_1,\omega_2)} = \int_{\omega_1}^{\omega_2} Q_\omega d\log\omega$

- Expanding in the symmetric mass ratio:

$$Q_\omega(\omega;\,\nu) = \underbrace{\frac{Q^0_\omega(\omega)}{\nu}}_{\text{0PA}} + \underbrace{Q^1_\omega(\omega)}_{\text{1PA}} + \underbrace{Q^2_\omega(\omega)\nu}_{\text{2PA}} + O(\nu^2)$$

fitting the coefficients
for a set of mass ratios
at fixed values of the frequency

THE COEFFICIENTS Q_ω^0, Q_ω^1, Q_ω^2



IMPROVING TEOBRESUMS

- The Q_ω analysis indicates that as the mass ratio increases, the dominant contributions are given by Q_ω^0 and Q_ω^1
- Q_ω^0 : depends on the 1st order self-force (1SF) flux
- Q_ω^1 : depends the 1SF and 2SF fluxes and on the 1SF contribution to the orbital potential
- Hence for higher mass ratios we have to improve TEOBResumS both in the conservative and in the dissipative sector of the model
- ... and we will turn off all NR calibration

GSF-INFORMED EOB POTENTIALS

$$A(u;\nu) = 1 - 2u + \nu a_{1\mathrm{SF}}(u) + \mathcal{O}(\nu^2)$$
$$\bar{D}(u;\nu) = \frac{1}{AB} = 1 + \nu \bar{d}_{1\mathrm{SF}}(u) + \mathcal{O}(\nu^2)$$
$$Q(u;\nu) = \nu q_{1\mathrm{SF}}(u) p_r^4$$

} EOB orbital potentials

Expressions for a_{1SF} , $\bar{d}_{1SF}$, q_{1SF} at 8.5PN order
+ suitable factorization & Padé-resummation
+ fit on the numerical GSF data of
Akcay & van de Meent, arXiv:1512.03392v2
… but singularity at the light-ring!

RADIATION REACTION (FLUX AT INFINITY)

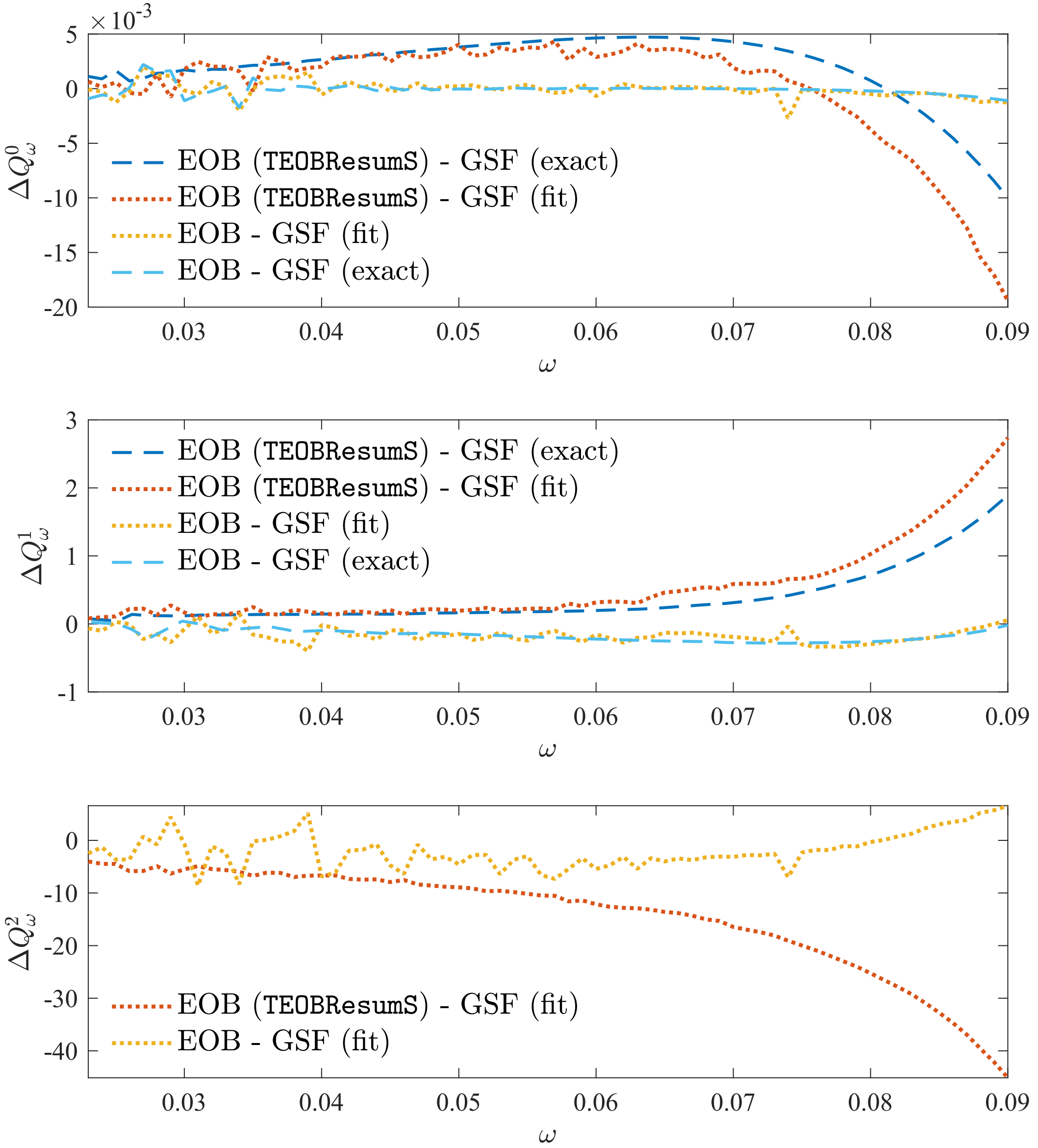
- Flux multipoles are factorized into different contributions, among which the residual amplitude corrections:

$$\rho_{\ell m} = \underbrace{1 + c_1 x + c_2 x^2 + \dots}_{\text{PN series}} \qquad x = \Omega^{2/3}$$

Ω → orbital frequency

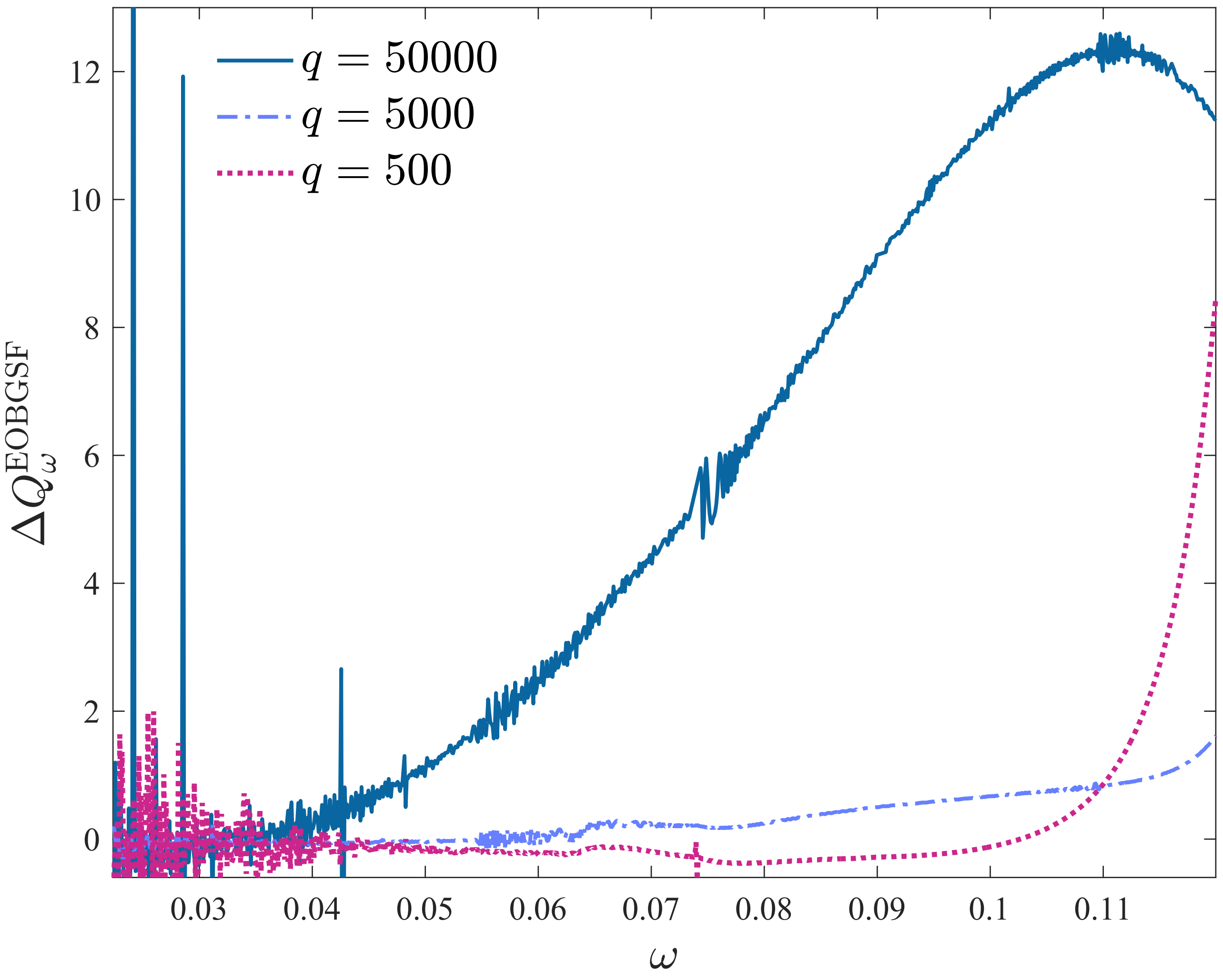
- The standard TEOBResumS has Padé-resummed 6PN expressions
- We hybridize 22PN results (Fujita 2012) with the known ν-dependence for every ℓm multipole (e.g. $c_1(\nu), c_2(\nu), c_3(\nu)$) up to $\ell = 8$

THE COEFFICIENTS Q_ω^0, Q_ω^1, Q_ω^2



Improved agreement in all the three functions. Also in Q_ω^2 due to having the EOB potentials only up to 1SF order now (no higher-order-in-ν corrections)

TOWARDS EXTREME-MASS-RATIO INSPIRALS



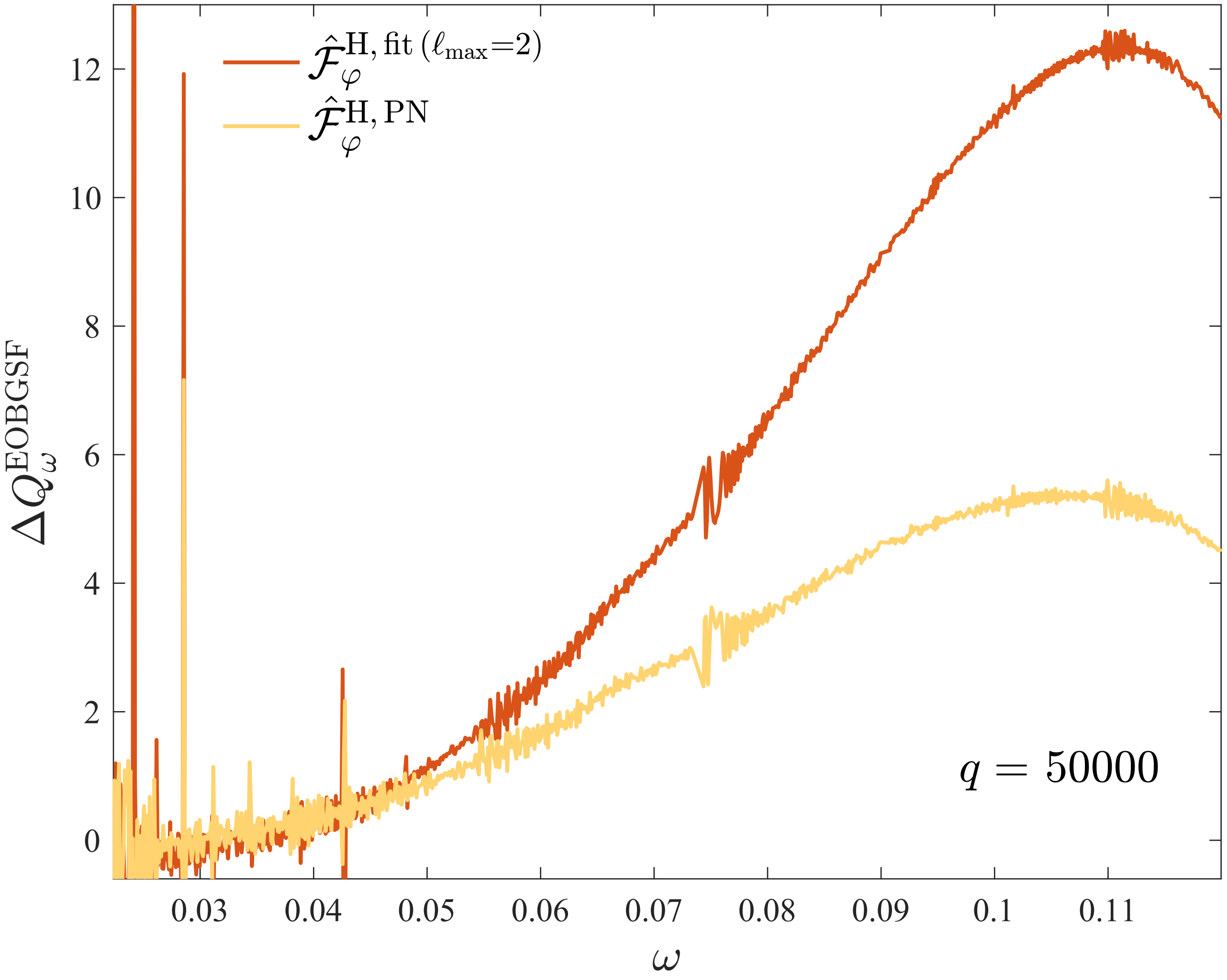
Integrated phase differences:

q = 500 $\Delta\phi \sim 0.07$

q = 5000 $\Delta\phi \sim 0.27$

q = 50 000 $\Delta\phi \sim 5.88$

THE HORIZON FLUX CONTRIBUTION

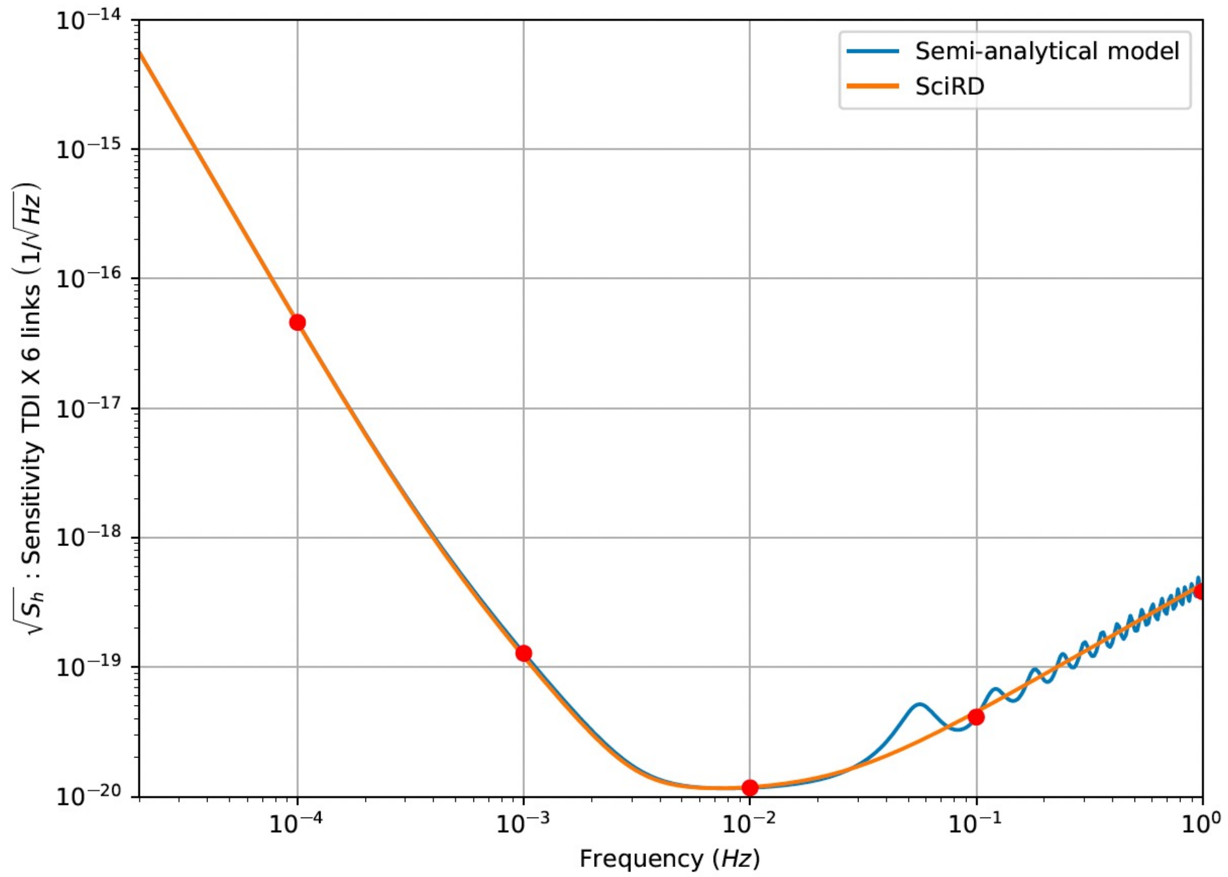


- To increase the agreement in $Q_\omega{}^0$: improving the horizon flux

- Integrated phase differences:

  Standard: $\Delta\phi \sim 5.88$
  Improved: $\Delta\phi \sim 2.94$

TALKING ABOUT LISA

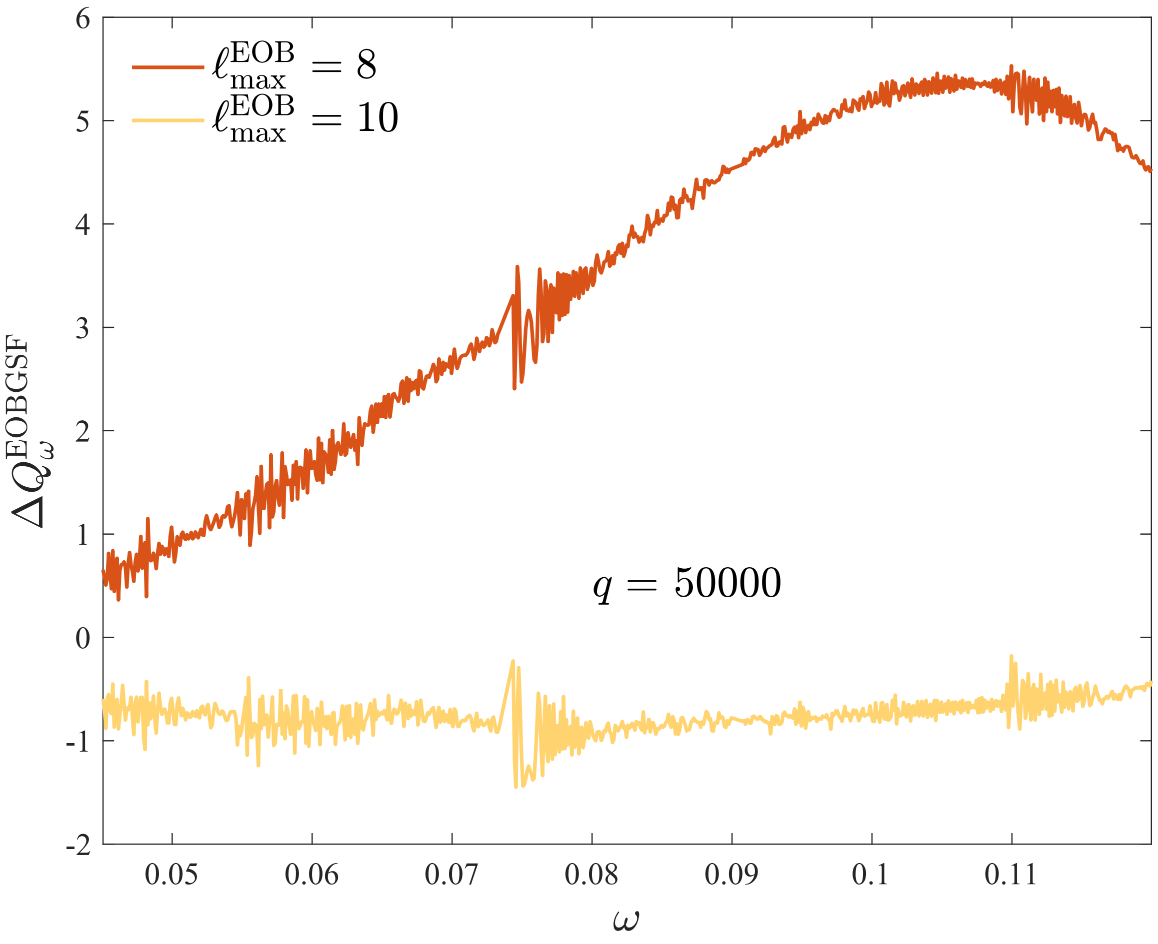


from Babak et al 2021, arXiv:2108.01167

So far we gained insight from the theorical point of view… but we should take into account:

- Frequency band where LISA will be sensitive
- Involved masses !
- Mission duration

… AND THINGS GET EVEN BETTER



- Adding ℓ = 9, 10 to the infinity flux
- Shorter frequency interval:

$\omega = [0.045, 0.12]$ $f = [0.003, 0.007]$ (Hz)

if $m_2 = 10 M_\odot$

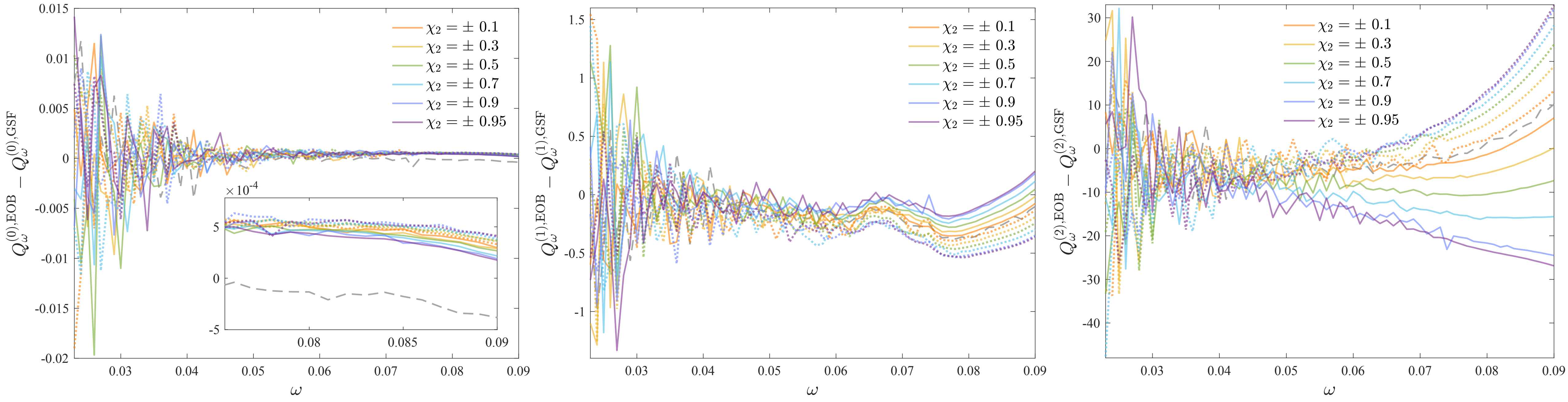
- Corresponding to ~1.2 years of EOB evolution, ~1.5 × 10^5 cycles
- Integrated phase differences:

Standard: $\Delta\phi \sim 2.99$
Improved: $\Delta\phi \sim -0.74$

SOME TECHNICAL DETAILS

- All the results shown here are obtained with the private MATLAB implementation of the code (quite slow)
- We have a public C implementation, that also exploits the post-adiabatic evolution (in EOB sense, see arXiv:1805.03891v2) during the inspiral. The GSF-informed potentials have been already implemented in the eccentric branch of the code, the new flux not yet!
- Other improvements:
  - different integration algorithm for the ODEs (maybe symplectic?)
  - eventually turning to machine learning for speed-up
  (e.g. see arXiv:2210.15684v2 for a frequency domain surrogate model for BNS based on TEOBResumS)

WORK IN PROGRESS: SPINNING SECONDARY



comparing $Q_\omega^{0,1,2}$ for different values of the secondary spin

WHY DO WE NEED BENCHMARKS?

- Not always for "calibration", but frequently just to make the right analytical choice:

χ_2	$\Delta\phi^{\rm EOBGSF}_{\rm DJS_{3.5PN}}$	$\Delta\phi^{\rm EOBGSF}_{\rm DJS_{4.5PN}}$	$\Delta\phi^{\rm EOBGSF}_{\rm antiDJS_{3.5PN}}$
0.5	−0.1037	−0.1025	−0.0886
−0.5	0.0477	0.047	0.0338
0.9	−0.1434	−0.1419	−0.0955
−0.9	0.1137	0.1116	0.0648

TABLE I. EOB − GSF phase difference evaluated via time-domain phasing for $q = 500$, either using the inverse resummation of G_{S_*} at 3.5PN or 4.5PN or its anti-DJS representation (Rettegno et al paper).

… and even different possibilities for the EOB spin-orbit sector could be explored

GENERIC DYNAMICS

- **Eccentricity**: could be switched on easily, currently exploring choices for the radiation reaction: arXiv:2104.10559v4, arXiv:2207.14002v1, arXiv:2305.19336v1 checked with respect to Teukolsky/RWZ solutions
- **Precession**: current version of TEOBResumS computes the evolution in the 'co-precessing' frame and then twists the waveform. Only spherical, and not tested for large mass ratios yet… Another possibility: Balmelli-Damour Hamiltonian (arXiv:1509.08135v1) for a real precessing evolution, but missing radiation reaction

BEYOND GR: SCALAR-TENSOR EOB

- Ongoing work in computing EOB quantities within massless scalar-tensor (ST) theories: see arXiv:2211.15580v2, arXiv:2301.01070, arXiv:2304.09052v1
  So far only conservative part of the dynamics: local-in-time and non-local-in-time ST corrections to the EOB potentials at 3PN + computation of the scattering angle

AND FINALLY… ENVIRONMENTAL EFFECTS

EOB is super flexible, so…

- Accretion (thin) disks: can be included in the flux
- Gravitating contribution: goes into the potentials
- Could also include a NS secondary
- Discussion:
  what's the most interesting thing we should include in the model?
  what kind of studies should we make?
  to which models should we compare ours then?

FYI: SCHWARZSCHILD + THIN DISK

Black hole encircled by a thin disk: fully relativistic solution*

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ABSTRACT

We give a full metric describing the gravitational field of a static and axisymmetric thin disk without radial pressure encircling a Schwarzschild black hole. The disk density profiles are astrophysically realistic, stretching from the horizon to radial infinity, yet falling off quickly at both these locations. The metric functions are expressed as *finite* series of Legendre polynomials. Main advantages of the solution are that (i) the disks have no edges, so their fields are everywhere regular (outside the horizon), and that (ii) *all* non-trivial metric functions are provided analytically and in closed forms. We examine and illustrate basic properties of the black-hole—disk space-times.

Kotlarik and Kofron (Institute of Theoretical Physics, Charles University in Prague) recently computed this exact solution in closed form! Could be useful…

CONCLUSIONS

- By having a benchmark at large mass ratios we are able to make the necessary modifications to TEOBResumS so as to make it useful for (quasi-circular nonspinning) EMRIs
- Still to do: improve the code and its speed, add several features, but we still need benchmarks, cannot be self-referential
- Open to collaborations to include various effects!