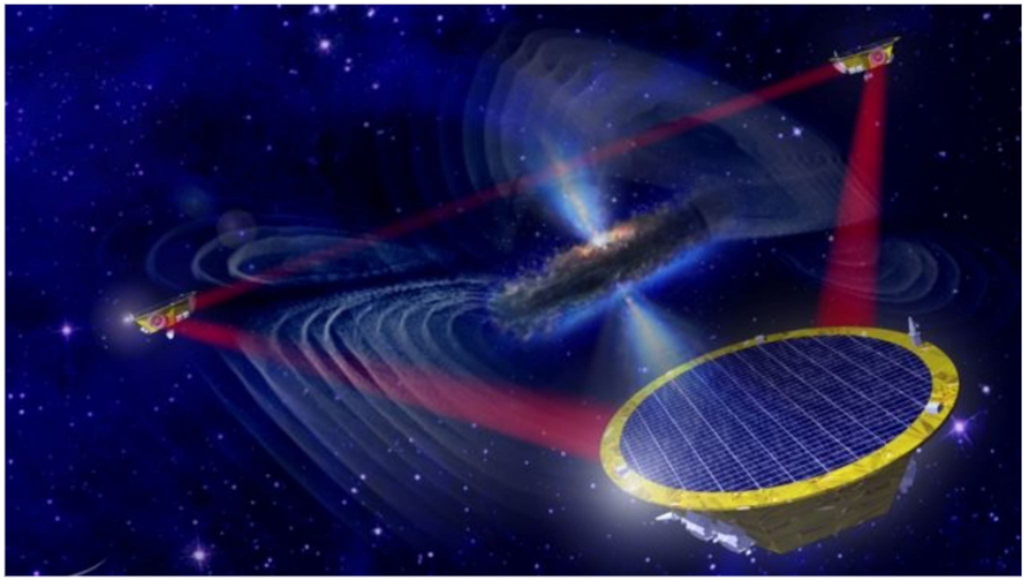
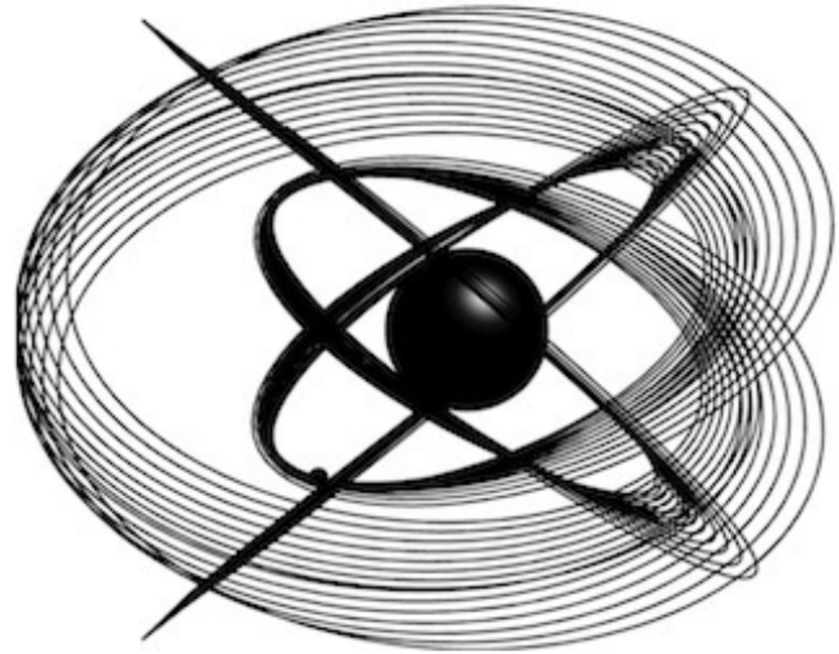
Extreme mass ratio inspiral modeling





How we develop waveform models for large mass ratio binaries: method, status, and thoughts on what is missing

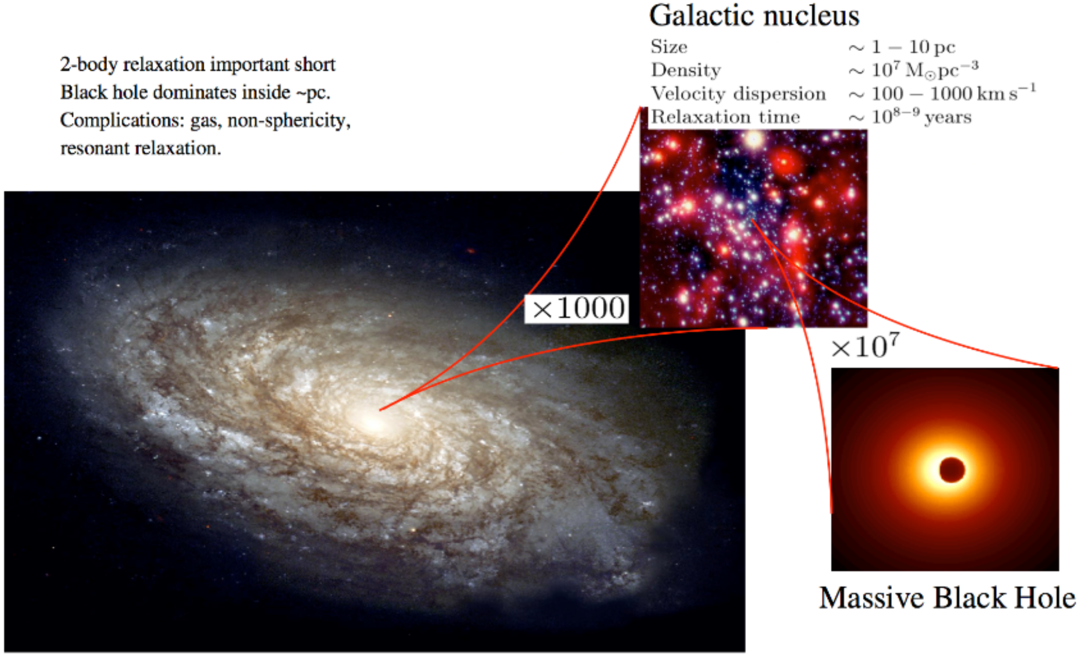
(Coordinated with Leor Barack's following presentation)

Scott A. Hughes, MIT

SISSA, 7 June 2023

Astrophysics overview

The setting: Center of a “normal” galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.

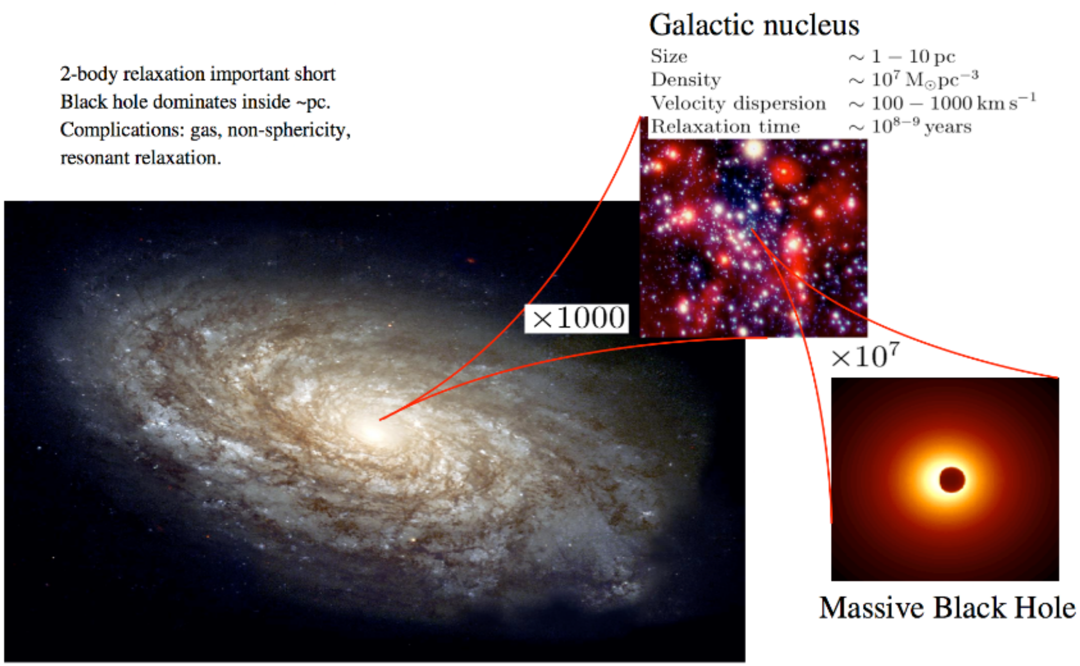


Graphic courtesy of Marc Freitag

The most massive of these stars tend to sink closest to the large black hole; these stars evolve through main sequence most quickly, will leave stellar mass black holes behind.

Astrophysics overview

The setting: Center of a "normal" galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.

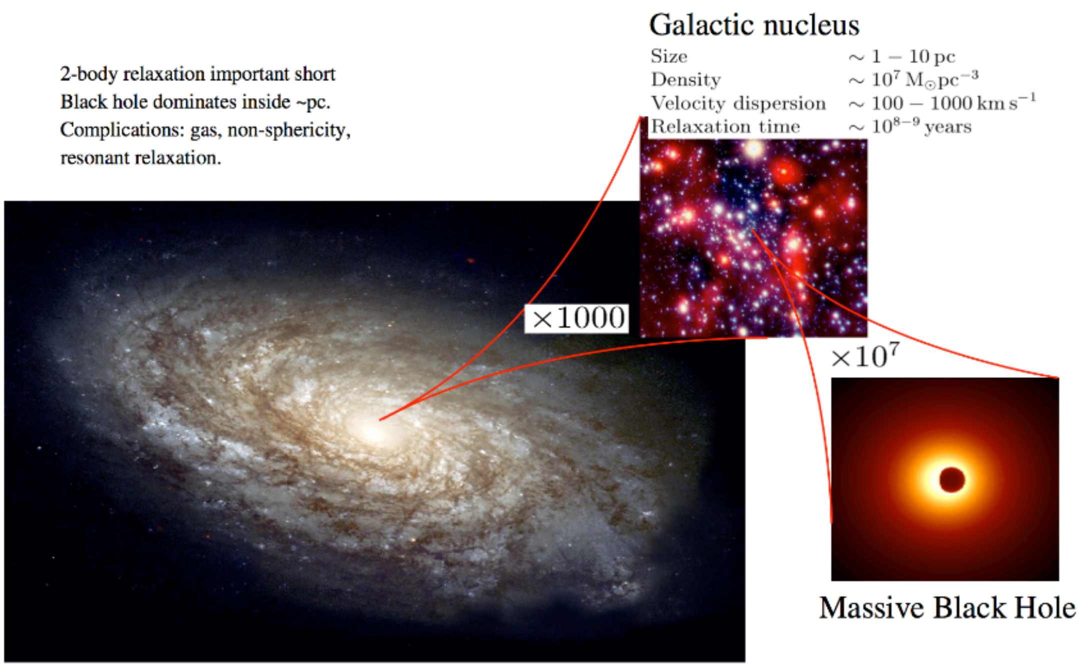


Graphic courtesy of Marc Freitag

Multi-body scattering in centers of galaxies puts compact stellar remnant onto an orbit that evolves into a strong-field, GW-driven inspiral.

Astrophysics overview

The setting: Center of a "normal" galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.



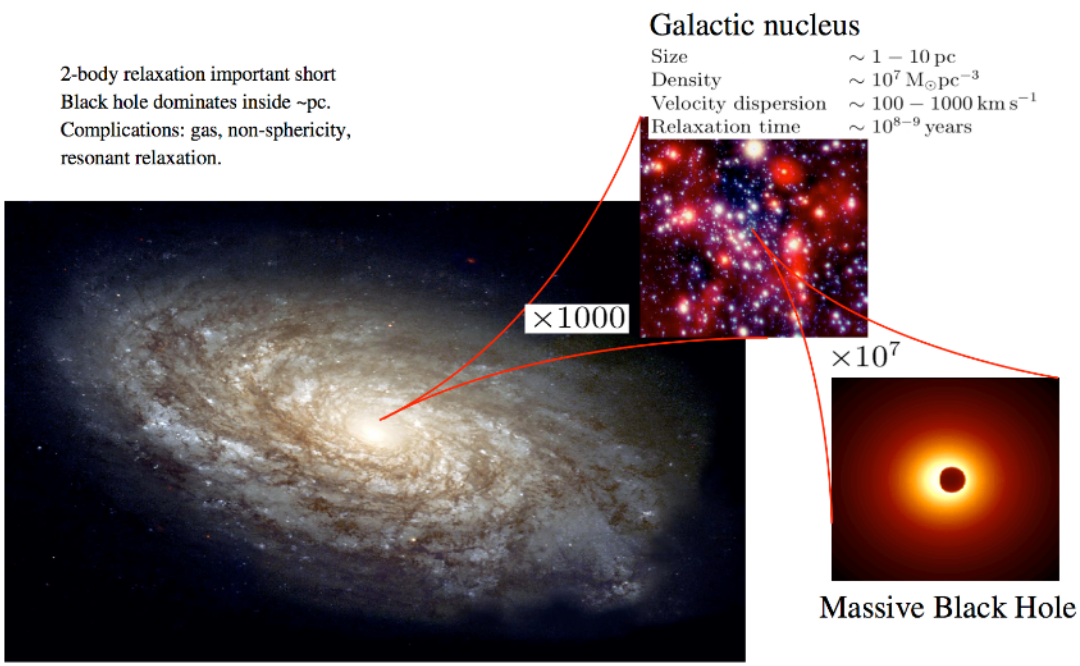
Graphic courtesy of Marc Freitag

Gravitational waves generated by these extreme mass ratio inspirals are in band

$$\frac{c^3}{50GM} \lesssim f \lesssim \frac{c^3}{GM}$$

Astrophysics overview

The setting: Center of a “normal” galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.



Graphic courtesy of Marc Freitag

Gravitational waves generated by these extreme mass ratio inspirals are in band

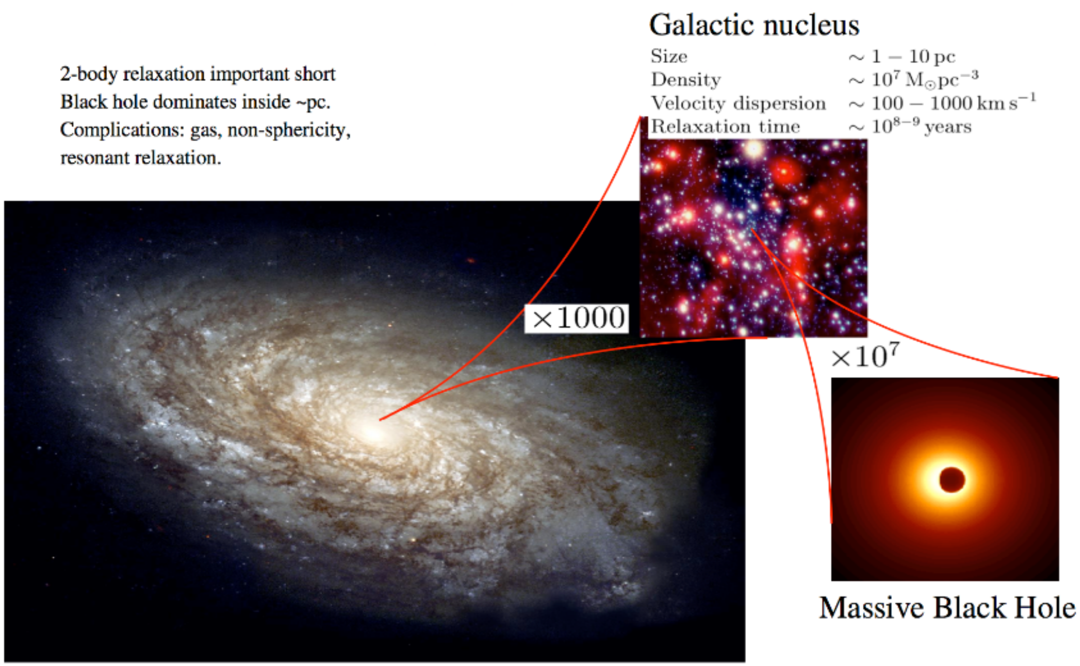
$$0.004\,\mathrm{Hz} \lesssim f \lesssim 0.2\,\mathrm{Hz}$$

at $M \approx 10^6$ Msun

Perfect for LISA!

Astrophysics overview

The setting: Center of a "normal" galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.

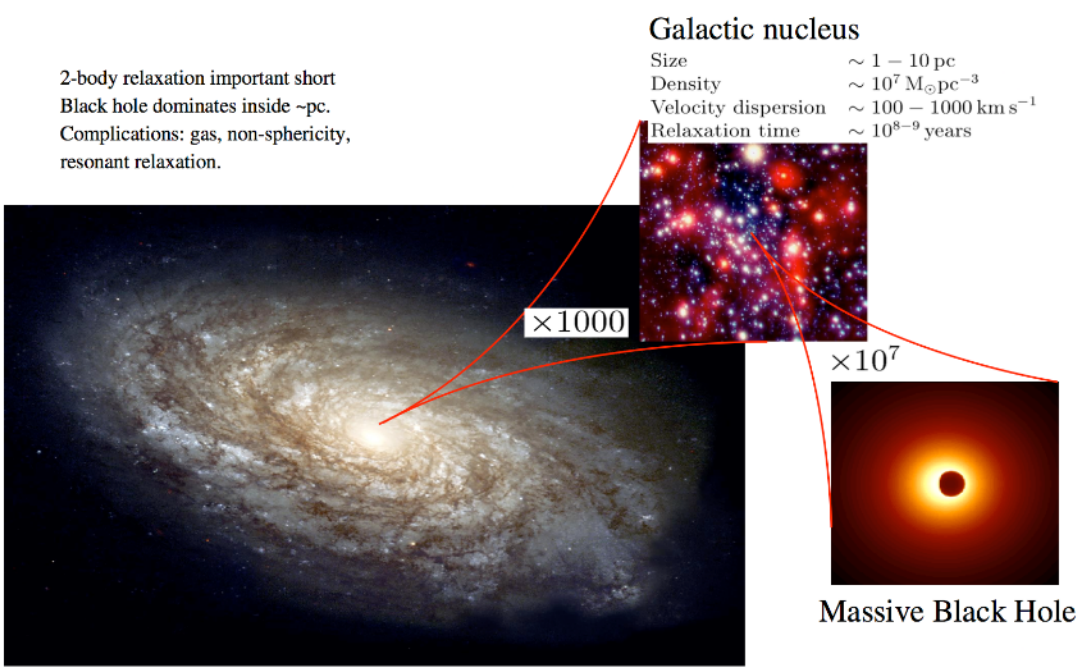


Graphic courtesy of Marc Freitag

Number of galaxies with the "right" central BHs plus studies of stellar scattering processes indicate event rate likely to be high: Perhaps > 10^2 per year if smaller object is 30 Msun or larger.

Astrophysics overview

The setting: Center of a “normal” galaxy. Typically hosts a black hole of 10^6—10^7 solar masses; black hole in a nucleus with ~10^9 solar masses of stars.



Graphic courtesy of Marc Freitag

Interesting related case: INTERMEDIATE mass ratio inspiral or IMRI. If smaller body is 10^2 or 10^3 Msun black hole, events are detectable to large z. **Likely an important fraction of early black hole mergers!**

The physics view of an EMRI

Get some intuition with leading order formulas:

Time spent spiraling from $f = f_1$ to $f = f_2$:

$$T_{\rm band} = \frac{5}{2^{2/3}\pi^{8/3}1024}\frac{c^3}{G\mu}\frac{c^2}{(GM)^{2/3}}\left({f_1}^{-8/3} - {f_2}^{-8/3}\right)$$

Months to years in band for $M \sim 10^6 - 10^7$ Msun, $\mu \sim 5 - 150$ Msun.

Number of orbits executed in that time:

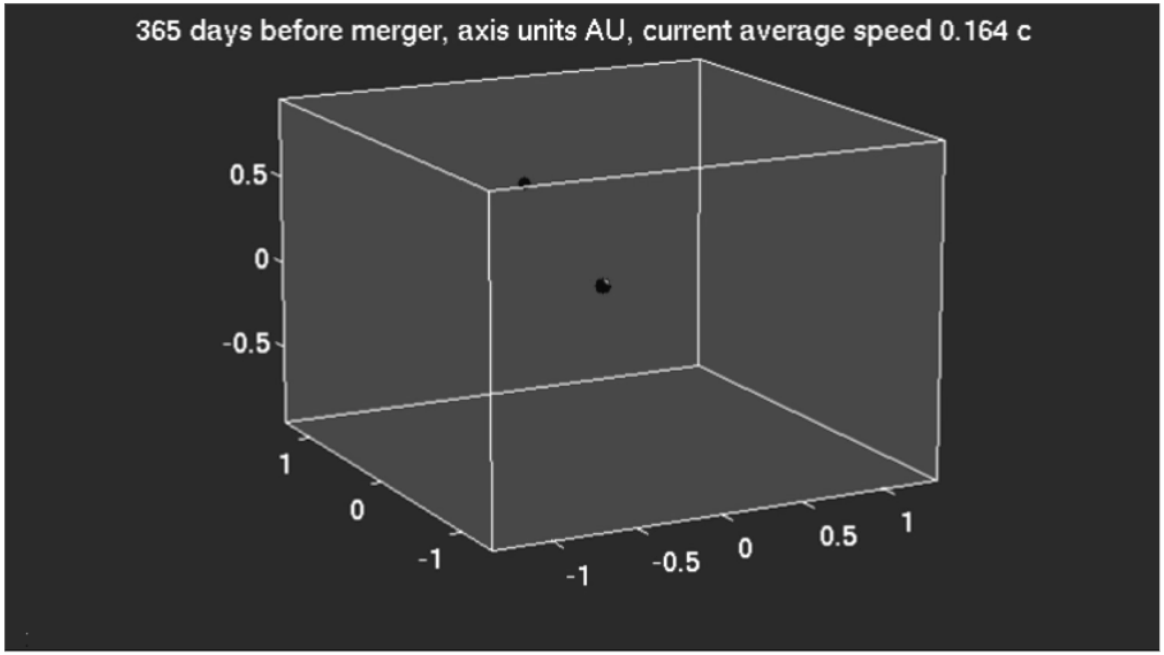
$$N_{\rm orb} = \frac{1}{2^{2/3}\pi^{8/3}256}\frac{c^3}{G\mu}\frac{c^2}{(GM)^{2/3}}\left({f_1}^{-5/3} - {f_2}^{-5/3}\right)$$

Tens of thousands of orbits executed during that time in band for these masses.

The physics view of an EMRI

Tens of thousands of slowly evolving orbits are executed in the near-field region of large black hole's spacetime … GWs that they generate are sensitive to the near-horizon black hole spacetime.

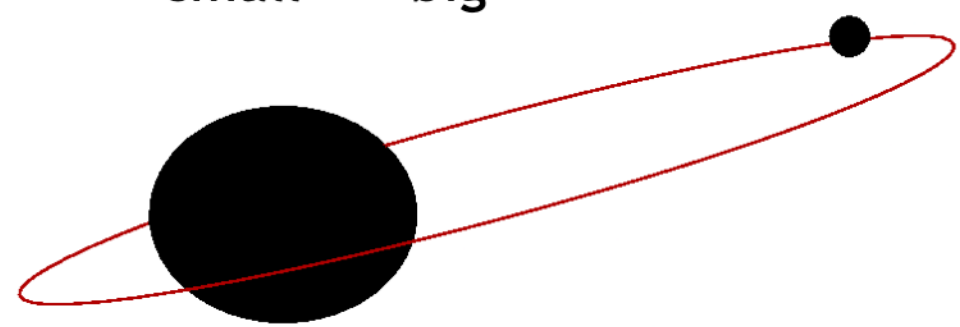
If we can coherently track these GWs, can use them to measure spacetime properties; expect measurement errors to scale as $1/N_{\rm orb}$ and 1/(signal to noise).



How we model these sources

Mass ratio serves as a natural perturbative expansion parameter: $m_{\text{small}}/M_{\text{big}} \ll 1$.

At least schematically, can write spacetime in the form



$$g_{\alpha\beta} = g^{\text{Kerr}}_{\alpha\beta}(M, a) + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta} + \dots$$

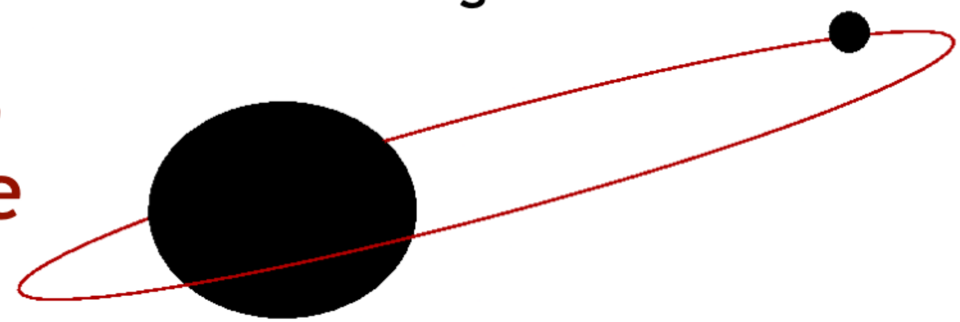
Motion of small body looks like a geodesic of g^{Kerr} plus corrections arising from perturbations $h^{(n)}$:

"Self force" f^{α} — correction to geodesic orbits due to h terms.

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = f^\alpha$$

How we model these sources

Mass ratio serves as a natural perturbative expansion parameter: $m_{\mathrm{small}}/M_{\mathrm{big}} \ll 1$.



Details are complicated. Eg, better to expand in curvature rather than metric:

$$\psi_n = \hat{\psi}_n + \delta\psi_n$$

Background

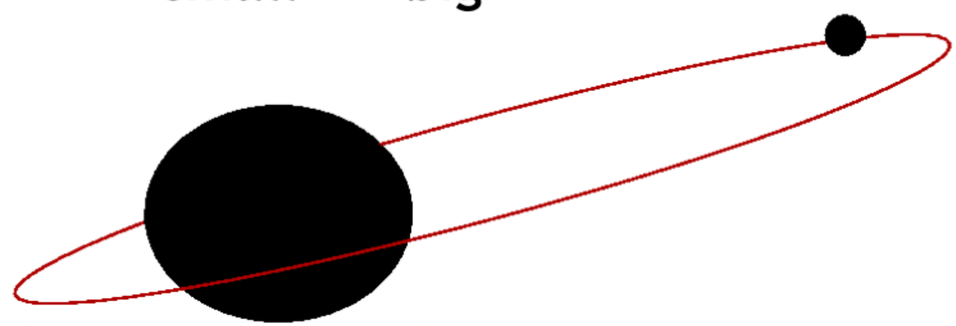
Perturbation

Newman-Penrose Weyl curvature scalar

For a Kerr black hole background, particularly important to get curvature components ψ_4 (ψ_0), which represent outgoing (ingoing) radiation … Teukolsky's perturbation equation gives an effective tool for this.

How we model these sources

Mass ratio serves as a natural perturbative expansion parameter: $m_{\mathrm{small}}/M_{\mathrm{big}} \ll 1$.



Once we have the curvature perturbation in hand, we know how to build metric perturbation (in some gauge) and construct the self force.

$$g_{\alpha\beta} = g^{\mathrm{Kerr}}_{\alpha\beta} + h^{(1)}_{\alpha\beta}(\psi_4, \psi_0) + \ldots$$

Once we have this (skipping a LOT of details!), can construct self force, examine motion of secondary and gravitational waves the system generates.

Motion: Action-angle approach

Following Flanagan and Hinderer [PRL **109**, 071102 (2012)], motion in BH spacetime with self force described using an action-angle framework:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

λ is a time variable (well adapted to Kerr orbits), and ε = *m*/*M*. Action and angle variables are

$$q_\alpha = (q_t, q_r, q_\theta, q_\phi) \qquad J_i = (E/m, L_z/m, Q/m^2)$$

Generalized angle variables describing motion in spacetime coordinates

Conserved quantities describing background motion

Motion: Action-angle approach

To understand motion in this framework, examine these equations order-by-order:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J}) + O(\varepsilon^2)$$

Motion: Action-angle approach

To understand motion in this framework, examine these equations order-by-order:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J})$$

$$\frac{dJ_i}{d\lambda} = 0$$

At order ε^0

0th order in mass ratio: Angle variables accumulate at rate set by their associated frequency; conserved quantities of background motion are conserved.

In simpler words: The zeroth order motion is a background spacetime geodesic.

Motion: Action-angle approach

To understand motion in this framework, examine these equations order-by-order:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon g_\alpha^{(1)}(q_\theta, q_r, \mathbf{J})$$

$$\frac{dJ_i}{d\lambda} = \varepsilon G_i^{(1)}(q_\theta, q_r, \mathbf{J})$$

At order ε^1

Very useful to understand these additional "forcing" terms by considering their Fourier expansions:

$$F(q_\theta, q_r) = \sum_{k,n} F_{kn} e^{i(kq_\theta + nq_r)}$$

$$F_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dq_\theta \int_0^{2\pi} dq_r\, F(q_\theta, q_r) e^{-i(kq_\theta + nq_r)}$$

Two timescales

Fourier expansion lets us identify terms that oscillate on short timescales and terms that accumulate secularly over longer intervals. For "most" orbits,

$$F(q_\theta, q_r) = \langle F \rangle + \delta F(q_\theta, q_r)$$

where

$$\langle F \rangle = F_{00} \qquad \delta F = \sum_{k,n \neq 0} F_{kn} e^{i(kq_\theta + nq_r)}$$

Accumulates over many orbits

Oscillates over an orbit, averages to (nearly) zero

Two timescales

The equations of motion then become

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha + \varepsilon\langle g_\alpha^{(1)}\rangle + \varepsilon\delta g_\alpha^{(1)}$$

$$\frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}\rangle + \varepsilon\delta G_i^{(1)}$$

Average of the forcing function $G^{(1)}{}_i$ describes the leading evolution of integrals of motion:

$$\langle G_i^{(1)}(\mathbf{J})\rangle = \left[\left\langle\frac{dE}{d\lambda}\right\rangle, \left\langle\frac{dL_z}{d\lambda}\right\rangle, \left\langle\frac{dQ}{d\lambda}\right\rangle\right]$$

Drives secular evolution of the system's orbital parameters, on time scale $M/\varepsilon = M^2/m$.

Two timescales

The equations of motion then become

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha + \varepsilon\langle g_\alpha^{(1)}\rangle + \varepsilon\delta g_\alpha^{(1)}$$

$$\frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}\rangle + \varepsilon\delta G_i^{(1)}$$

Average of the forcing function $g^{(1)}{}_\alpha$ equivalent to a shift in the orbital frequencies:

$$\omega_\alpha(\mathbf{J}) \to \omega_\alpha(\mathbf{J}) + \varepsilon\langle g_\alpha^{(1)}(\mathbf{J})\rangle$$

Leading conservative impact on the binary: Frequencies are shifted of order $\varepsilon = m/M$.

Summary: Two-timescale character of EMRI evolution

We can now catalog the various terms and their impact on EMRIs at 1st order in mass ratio:

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha + \varepsilon\langle g_\alpha^{(1)}\rangle + \varepsilon\delta g_\alpha^{(1)}$$

Geodesic freq., ~$1/M$

Leading conservative correction, ~m/M^2

Oscillatory conservative contribution

$$\frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}\rangle + \varepsilon\delta G_i^{(1)}$$

Leading dissipation, changes orbit on $T_r \sim M^2/m$

Oscillatory contribution to the dissipation

How phase accumulates

Examine how different effects accumulate if we follow an EMRI's phase over a time interval.

Phase accumulated looks like frequencies (and corrections) integrated over interval determined by dissipation:

Phase accumulated from t_1 to t_2: $$\Phi(t_1, t_2) = \int_{t_1}^{t_2} \omega(t)\, dt$$

O(*M*/*m*): Evolving geodesic freq. [O(1/*M*)] integrated over insp. [O(M^2/m)] → $= \Phi_{\mathrm{diss}-1}$

O(1): Cons. correction to freq. [O(m/M^2)] integrated over inspiral → $+ \Phi_{\mathrm{cons}-1}$

O(1): Geodesic freq. integrated against next order correction to inspiral [O(*M*)] → $+ \Phi_{\mathrm{diss}-2}$

O(*m*/*M*): next order correction to freq. [O(m^2/M^3)] integrated against inspiral [O(M^2/m)] → $+ \Phi_{\mathrm{cons}-2}$

$$+ \ldots$$

How phase accumulates

Conventional wisdom: Detecting GWs requires models accurate to O(1) on phase.

Phase accumulated looks like frequencies (and corrections) integrated over interval determined by dissipation:

Phase accumulated from t_1 to t_2: $$\Phi(t_1, t_2) = \int_{t_1}^{t_2} \omega(t)\, dt$$

O(*M*/*m*): Evolving geodesic freq. [O(1/*M*)] integrated over insp. [O(M^2/m)] → $= \Phi_{\mathrm{diss}-1}$

O(1): Cons. correction to freq. [O(m/M^2)] integrated over inspiral → $+ \Phi_{\mathrm{cons}-1}$

O(1): Geodesic freq. integrated against next order correction to inspiral [O(*M*)] → $+ \Phi_{\mathrm{diss}-2}$

O(*m*/*M*): next order correction to freq. [O(m^2/M^3)] integrated against inspiral [O(M^2/m)] → $+ \Phi_{\mathrm{cons}-2}$

$+ \ldots$

Where does modeling stand today?

For ***detection only***, arguments based on phase counting and conventional wisdom tell us

$$\Phi_{\rm needed} = \Phi_{\rm diss-1} + \Phi_{\rm cons-1} + \Phi_{\rm diss-2}$$
$$= \Phi_{\rm 0PA} \quad + \quad \Phi_{\rm 1PA}$$

From 1st order averaged dissipative self force.
Well understood: Used for "adiabatic" waveforms

From rest of the 1st order self force.
Understood; expensive & challenging to compute.

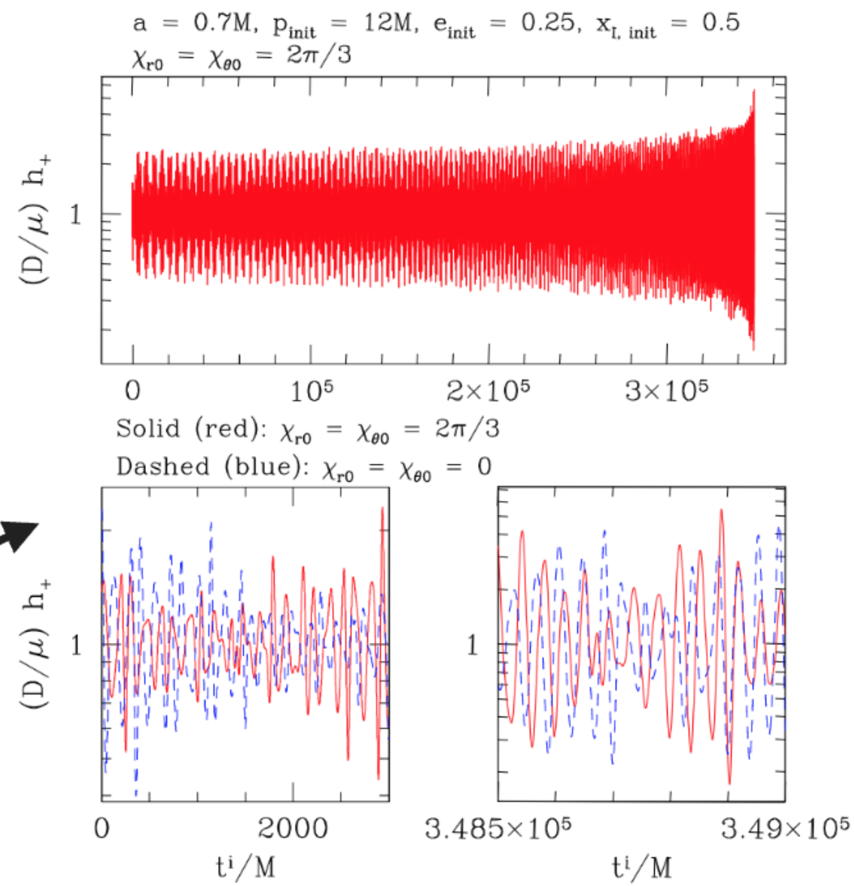
From 2nd order self force:
Frontier: 1st results recently published.

Leading order: Adiabatic waveform (OPA)

Leading approximation treats the motion as geodesic on short timescales, "flows" through sequence of geodesics due to GW backreaction.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle$$

This approximation yields the leading contribution to orbital phase, scaling as *M/m* (ie, 1/mass-ratio).



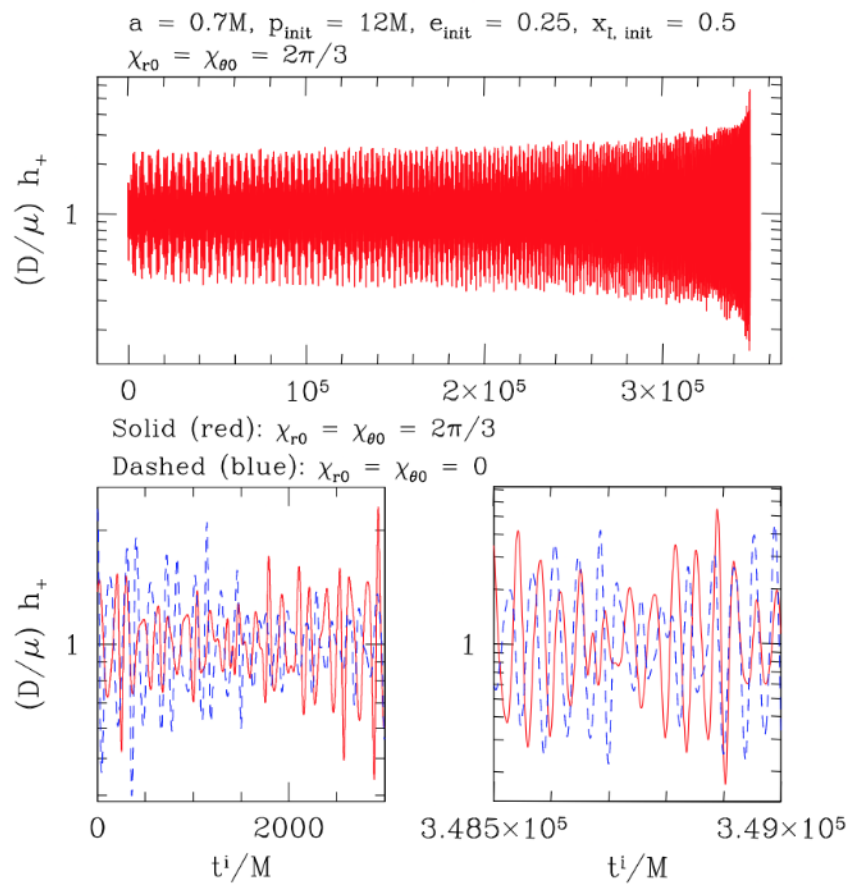
Example adiabatic waveform, Fig. 16 of Hughes et al PRD 2021 (arXiv:2102.02713)

Leading order: Adiabatic waveform (OPA)

Leading approximation treats the motion as geodesic on short timescales, "flows" through sequence of geodesics due to GW backreaction.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle$$

Serves as the foundation for FEW (Fast EMRI Waveforms); not difficult to fold in leading conservative effects once we have a deep catalog of results to draw on.

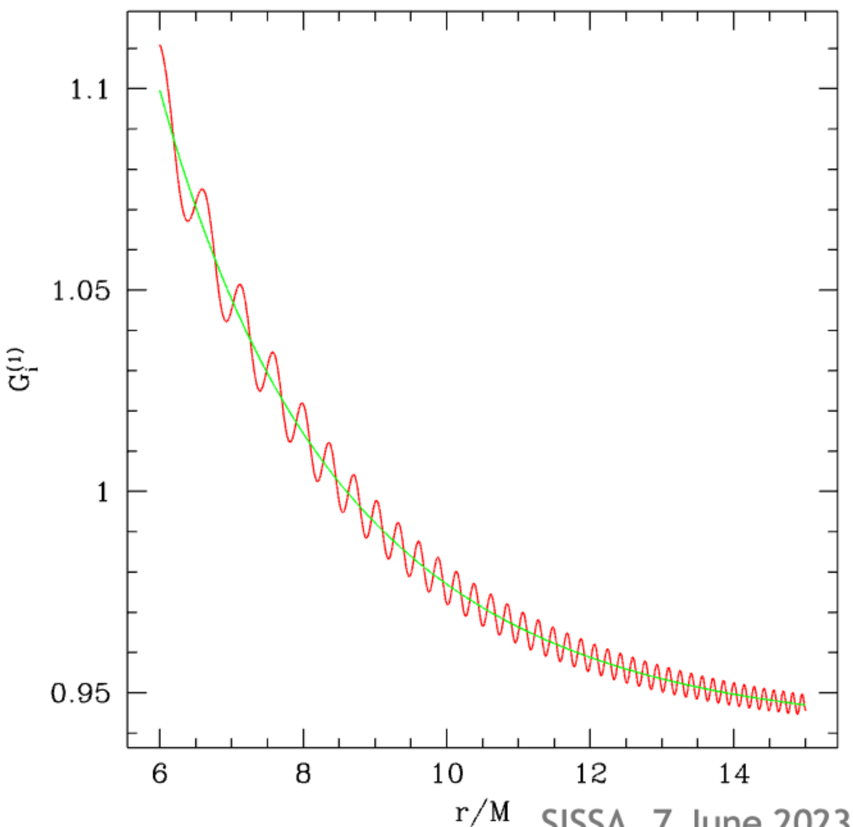


Leading order: Adiabatic waveform (OPA)

Leading approximation treats the motion as geodesic on short timescales, "flows" through sequence of geodesics due to GW backreaction.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle$$

Why this approximation works: for "most" orbits, self force components dominated by average values. Neglecting oscillations is acceptable as a first approximation for inspiral.

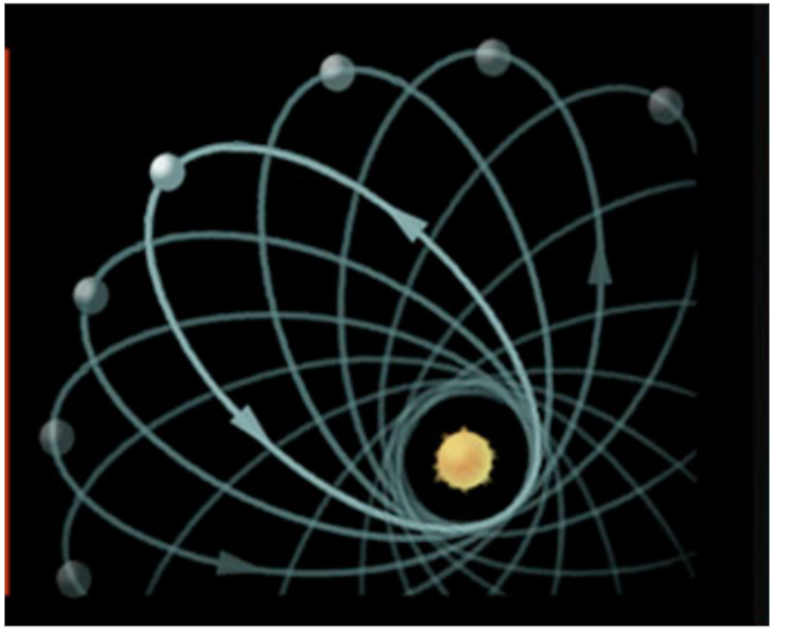


Leading order: Adiabatic waveform (OPA)

Leading approximation treats the motion as geodesic on short timescales, "flows" through sequence of geodesics due to GW backreaction.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) + \varepsilon\langle g_\alpha^{(1)}\rangle \qquad \frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle$$

Example important 1PA term: conservative contribution to frequencies. Straightforward to include once we have a good "catalog" of its behavior over parameter space.



Example effect of such a term: Small body's own contribution to the rate of periapsis precession.

Leading order: Adiabatic waveform (OPA)

Leading approximation treats the motion as geodesic on short timescales, "flows" through sequence of geodesics due to GW backreaction.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J}) \qquad \frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle + \varepsilon^2\langle G_i^{(2)}\rangle$$

Example important 1PA term:
Next order dissipative contribution to inspiral rate.
Requires next order in perturbation theory!
Frontier of current research.

See Leor Barack's talk for details and discussion of current status

Other physics: secondary spin

Most of this discussion imagines the smaller body is a structureless point … if it is itself a Kerr black hole, then we need to account for its spin.

0th order motion is not a geodesic in this case …

$$\frac{Dp^\alpha}{d\tau} = 0$$

Other physics: secondary spin

Most of this discussion imagines the smaller body is a structureless point … if it is itself a Kerr black hole, then we need to account for its spin.

0th order motion is not a geodesic in this case … but is instead governed by the Mathisson-Dixon-Papapetrou equations of motion: schematically, a forced geodesic coupled to spin precession.

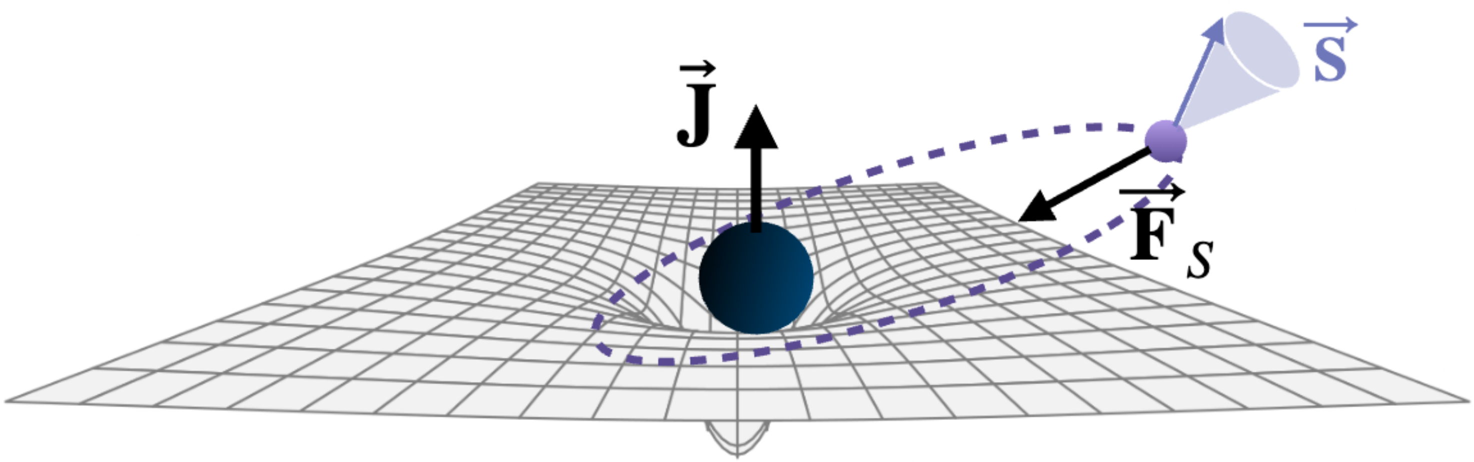
$$\frac{Dp^\alpha}{d\tau} = -\frac{1}{2} R^\alpha{}_{\beta\mu\nu} u^\beta S^{\mu\nu}$$

$$\frac{DS^{\mu\nu}}{d\tau} = p^\mu u^\nu - p^\nu u^\mu$$

$$p_\mu S^{\mu\nu} = 0$$

Other physics: secondary spin

Most of this discussion imagines the smaller body is a structureless point … if it is itself a Kerr black hole, then we need to account for its spin.



Precession and spin-curvature coupling force change orbital frequencies and orbit properties … effect comparable to leading self force. Now updating adiabatic framework to include these effects.

Can we fold accretion physics into this framework?

Accretion has the potential to change the orbits that a small body follows, as well as the backreaction / inspiral / wave emission.

Kocsis, Yunes, Loeb approach: Leave orbits essentially unchanged, but add evolutionary terms accounting for accretion-induced sinks of energy and angular momentum.

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J})$$

$$\frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle + F_i^{\mathrm{acc}}(\mathbf{J})$$

Can we fold accretion physics into this framework?

Accretion has the potential to change the orbits that a small body follows, as well as the backreaction / inspiral / wave emission.

Guidance needed here!

$$\frac{dq_\alpha}{d\lambda} = \omega_\alpha(\mathbf{J})$$

$$\frac{dJ_i}{d\lambda} = \varepsilon\langle G_i^{(1)}(\mathbf{J})\rangle + F_i^{\rm acc}(\mathbf{J})$$