Black holes, gluing, and all that

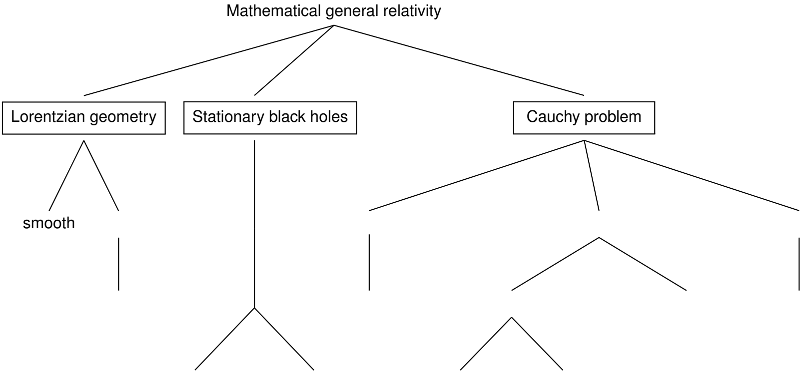
Piotr T. Chruściel

University of Vienna

Trieste, SISSA, July 2023

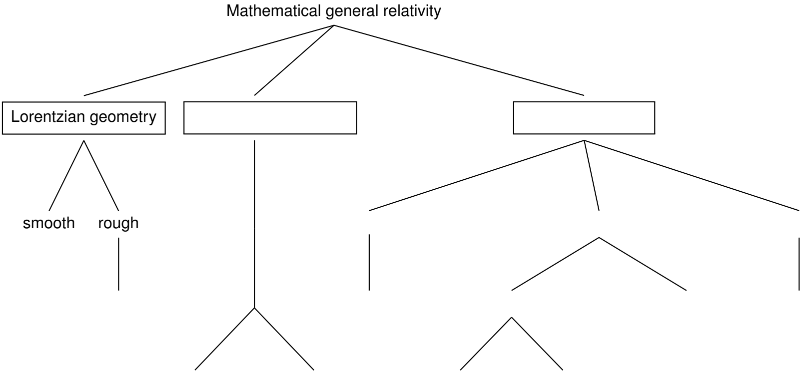
The landscape of Mathematical General Relativity

Penrose, Hawking, Galloway, Beem, Ehrlich, Minguzzi,...



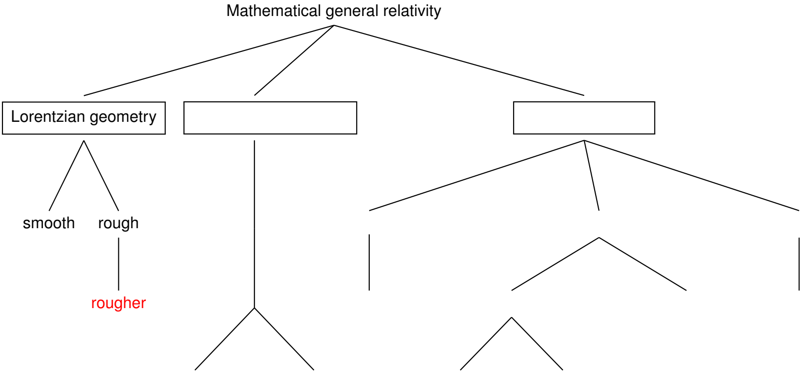
The landscape of Mathematical General Relativity

Sbierski; Klainerman-Szeftel; Graf, Kunzinger, Ohanyan, Steinbauer,



The landscape of Mathematical General Relativity

Kunzinger, Ohanyan, Steinbauer, Sämann, McCann, Cavaletti, Mondino, ...



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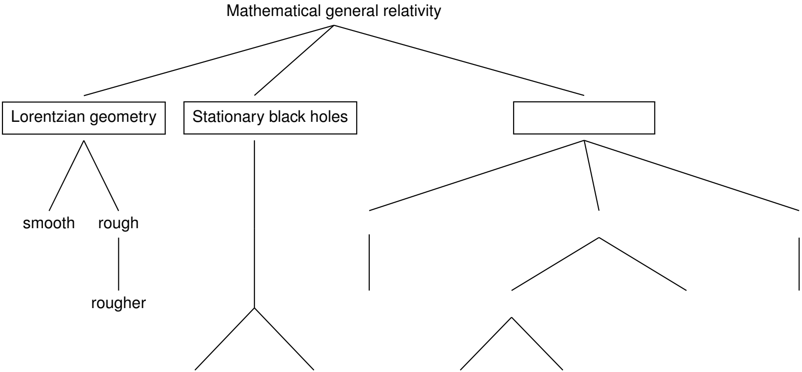
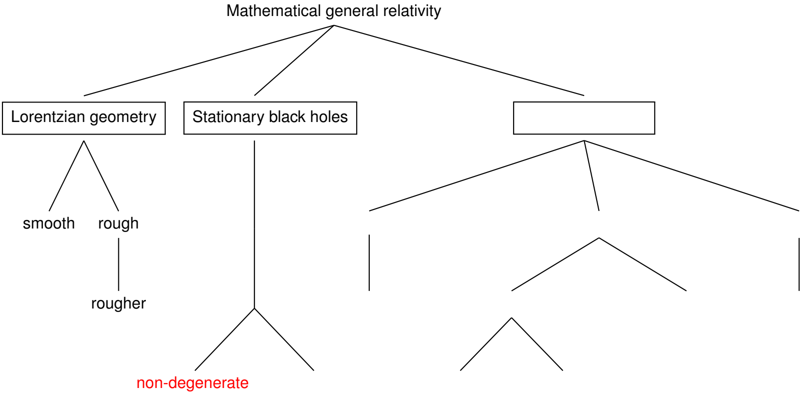
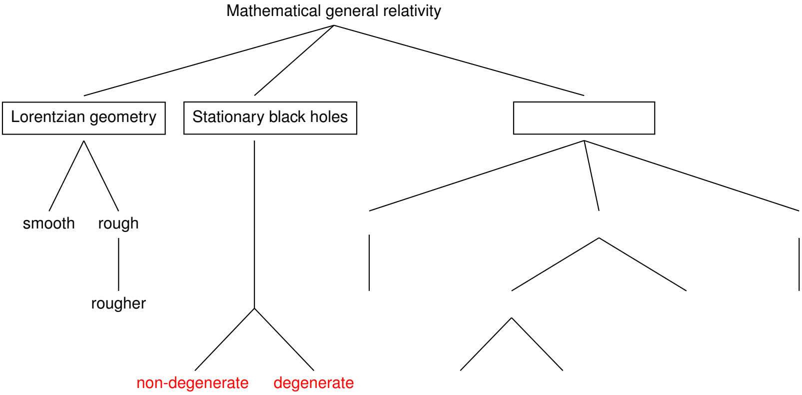




Mathematical general relativity
Lorentzian geometry
Stationary black holes
smooth
rough
rougher
non-degenerate



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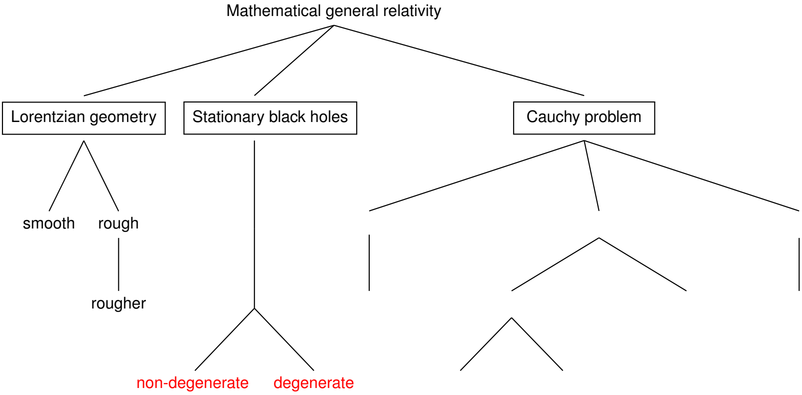

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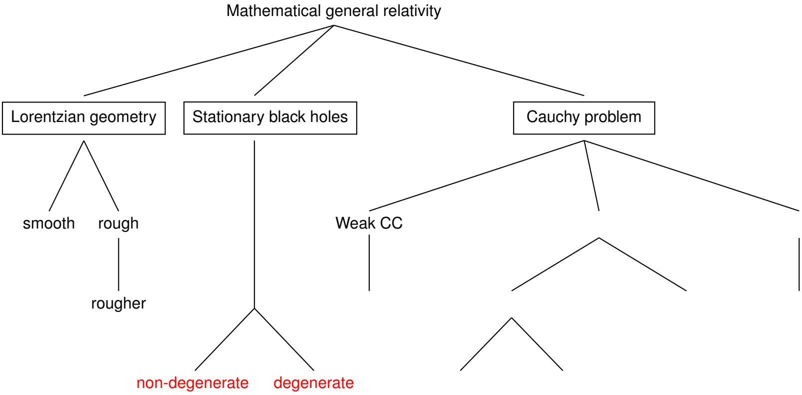
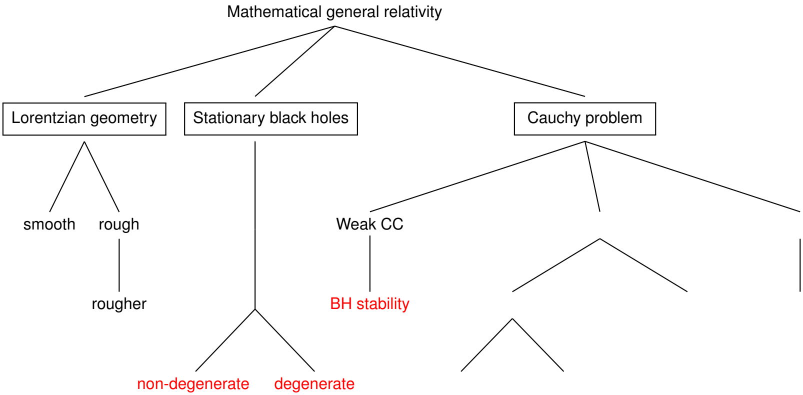
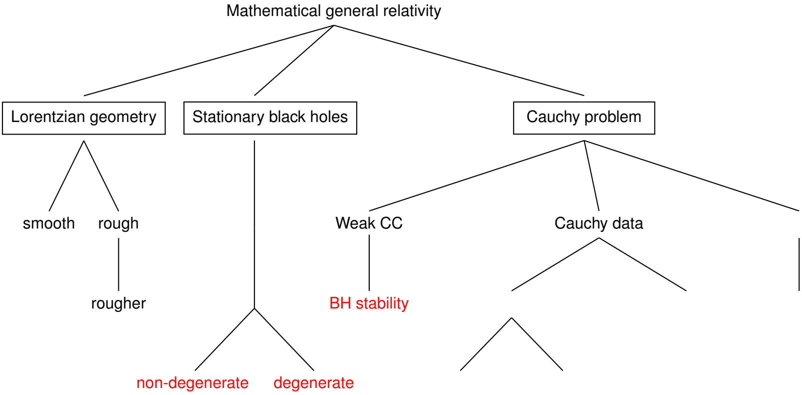


The landscape of Mathematical General Relativity

many authors: Klainerman, Giorgi, Szeftel, Dafermos, Holzegel, Taylor, Hintz, Vasy, Andersson, Blue, Ma, Moschidis, ...



The landscape of Mathematical General Relativity



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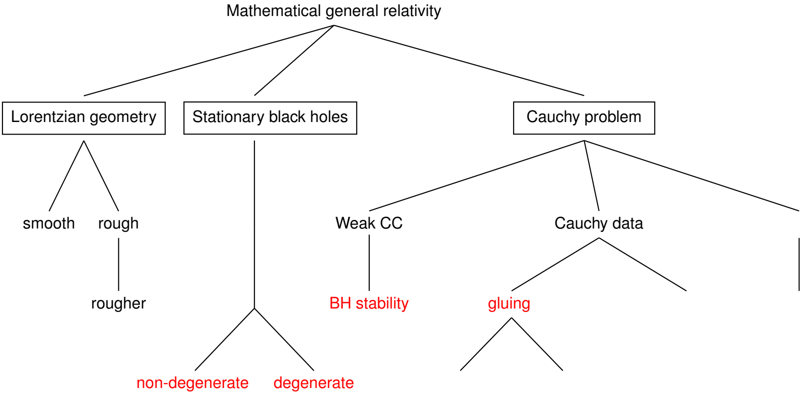
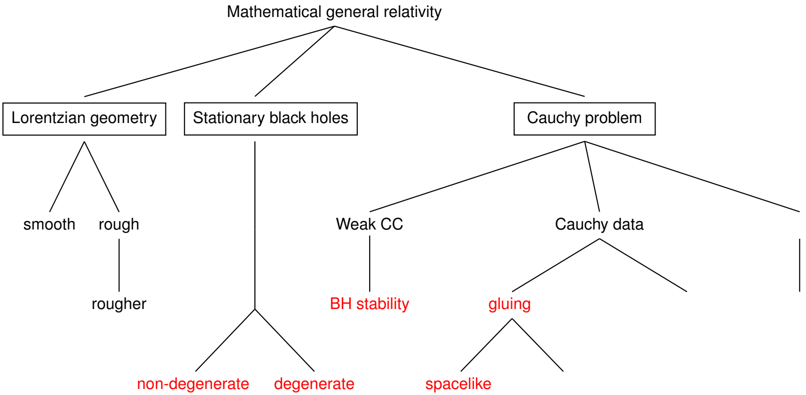


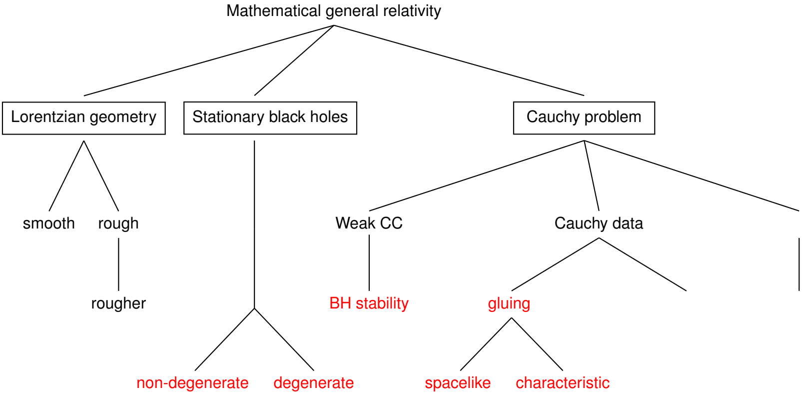
The landscape of Mathematical General Relativity

Corvino & Schoen, Carlotto & Schoen, PTC & Delay, Delay & Mazzieri, Isenberg, Lee & Stavrov, Czimek, Mao, Oh, Tao, ...



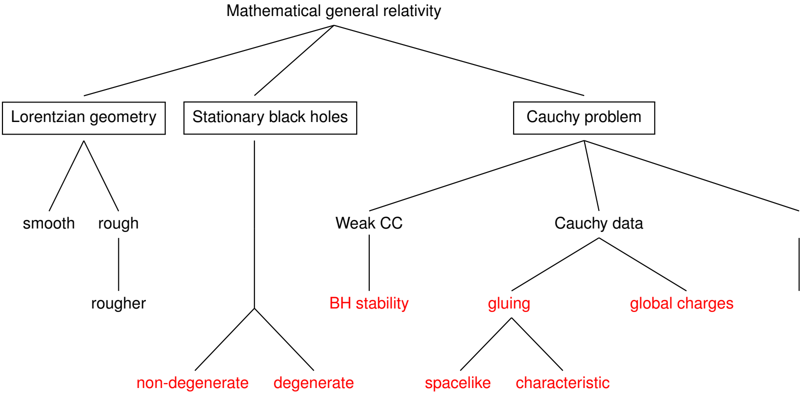
The landscape of Mathematical General Relativity

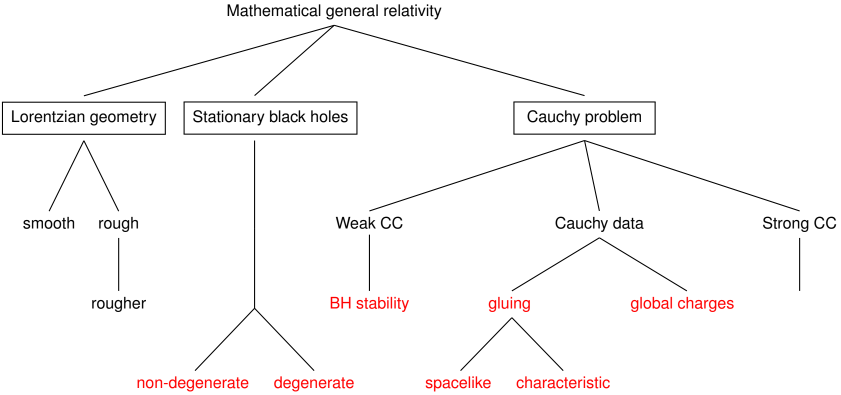
Aretakis, Czimek & Rodnianski, Kehle & Unger, PTC, Cong & Gray



The landscape of Mathematical General Relativity

many authors, Benatti, Fognagnolo & Mazzieri




Mathematical general relativity
Lorentzian geometry
Stationary black holes
Cauchy problem
smooth
rough
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Weak CC
Cauchy data
Strong CC
BH stability
gluing
global charges
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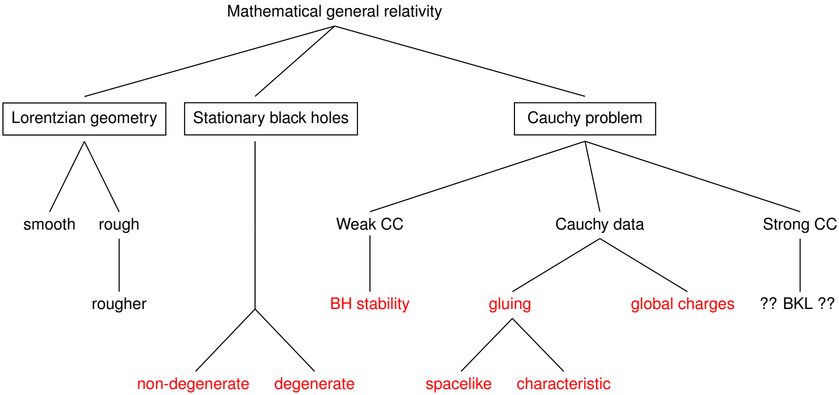
degenerate
spacelike
characteristic

The landscape of Mathematical General Relativity

Belinski, Khalatnikov, Liftshitz: generic solutions of Einstein equations behave chaotically near singularities?



Black holes?

Throughout this talk: vacuum spacetimes with $\Lambda \in \mathbb{R}$

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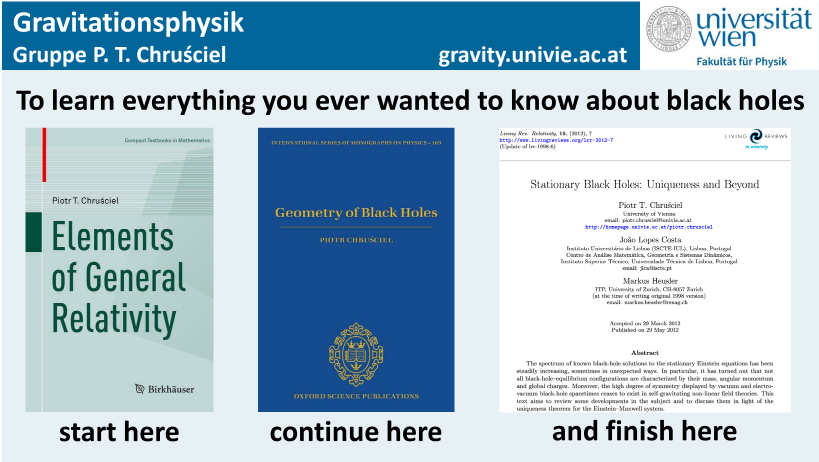

Static vs. stationary

Time-independent can be static or stationary;

- *static:* stationarity plus *time-reversal isometry*
- *Regular*, static, black hole *exteriors* $(\mathcal{M}, g)$ take the form $\mathcal{M} = \mathbb{R} \times \Sigma$,

$g = -V^2 dt^2 + \gamma$, and the Riemannian metric γ satisfies

$$\textit{in vacuum}: \quad V \mathrm{Ricci}(\gamma) = \mathrm{Hess}\, V \;, \; \Delta V = 0 \;,$$

with $\partial\Sigma = \{V = 0\}$. $\partial\Sigma$ = non-degenerate horizons; asymptotically cylindrical ends = degenerate horizons

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Static vs. stationary and degenerate vs. non-degenerate

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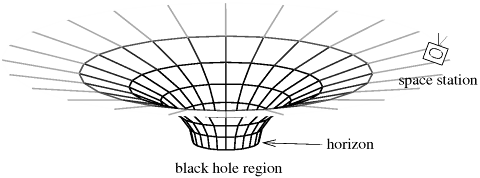

Embedding a *non-degenerate* space-geometry in higher dimension

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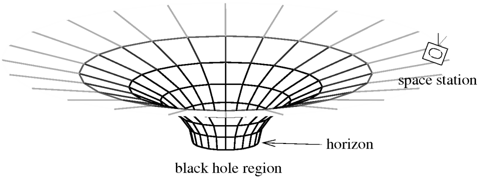

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$\Lambda = 0$: "Black Holes have No Hair"

The **analytic, connected** classification in space-time dimension **four**; contributions by Israel, Hawking, Carter, Robinson, Bunting, Mazur, PTC-Costa Lopes, PTC-Sudarsky-Wald,

Stationary,
vacuum,
analytic,
connected,
regular black hole
=
Kerr-Newman

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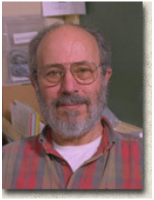

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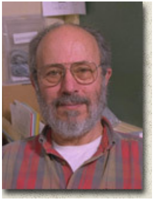

Stationary,
electro-vacuum,
analytic,
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=
Kerr-Newman
in the exterior region

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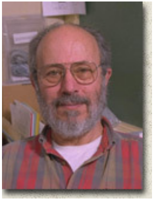
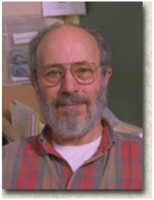


Stationary,
electro-vacuum,
analytic,
connected,
regular black hole
=
Kerr-Newman
in the exterior region

Static,
electro-vacuum,
regular black hole
=
MP or RN
in the exterior region

$\Lambda = 0$: "Black Holes have No Hair"

The **analytic, connected** classification in space-time dimension **four**; contributions by Israel, Hawking, Carter, Robinson, Bunting, Mazur, PTC-Costa Lopes, PTC-Sudarsky-Wald, Masood-ul-Alam, Ruback, PTC-Galloway, PTC-Reall-Tod, new proof by Agostini-Mazzieri



Stationary, electro-vacuum, analytic, connected, regular black hole = **Kerr-Newman** in the exterior region	Static, electro-vacuum, regular black hole = **MP or RN** in the exterior region

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Theorem (announced by Klainerman, Giorgi & Szeftel (2021, 2022))

Near-Schwarzschild non-degenerate vacuum black holes with $\Lambda = 0$ *evolve to Kerr*

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Near-Schwarzschild non-degenerate vacuum black holes with $\Lambda = 0$ *evolve to Kerr*

- The end state has no reason to be analytic, and therefore a uniqueness theorem assuming analyticity is nice but not useful

Analyticity?

Some old progress: Alexakis, Ionescu, Klainerman arXiv:0904.0982 [gr-qc]

Theorem (Alexakis, Ionescu, Klainerman (2009))

Regular non-degenerate stationary vacuum black holes near non-extreme Kerr are Kerr

Analyticity?

Some old progress: Alexakis, Ionescu, Klainerman arXiv:0904.0982 [gr-qc]

Theorem (Alexakis, Ionescu, Klainerman (2009))

Regular non-degenerate *stationary vacuum black holes near non-extreme Kerr are Kerr (no assumption of analyticity)*

Static vacuum field equations

with cosmological constant, space dimension n, normalised

$$g = -V^2 dt^2 + g_{ij} dx^i dx^j \,, \qquad \partial_t V = 0 = \partial_t g_{ij}$$

$$V R_{ij} + D_i D_j V = \pm n V g_{ij} \,,$$

$$\Delta V = \mp n V$$

Known solutions: Birmingham-Kottler (Schwarzschild-de Sitter):

$$g = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \qquad V^2 = 1 - \frac{\Lambda r^2}{3} - \frac{2m}{r} \,.$$

$m \in \mathbb{R}$, or Nariai

$$g = -(\lambda - \Lambda r^2) dt^2 + \frac{dr^2}{\lambda - \Lambda r^2} + |\Lambda|^{-1} h_\kappa$$

$\kappa = \pm 1$, $\kappa\Lambda > 0$, $\lambda \in \mathbb{R}$

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$\Lambda > 0$: stationary, vacuum, close to Schwarzschild-de Sitter

Hintz (2017)

$$g = -V^2(dt + \theta_i dx^i)^2 + g_{ij}dx^i dx^j \,, \qquad \partial_t V = 0 = \partial_t g_{ij} = \partial_t \theta_i \,.$$

Theorem

Stationary solutions close to Schwarzschild-de Sitter are the slowly rotating Kerr-de Sitter metrics.

The proof builds on the proof of *dynamical stability of the region between horizons of slowly rotating KdS spacetimes* by Hintz & Vasy (2016)

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$\Lambda > 0$: Uniqueness, static case

only partial results

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Theorem (Borghini & Mazzieri & PTC 2021)

Schwarzschild-de Sitter's are the only static black holes

with an ombilical & separating maximal level set of V

and satisfying a "virtual mass" condition

(special case of more general theorems); builds on previous work by Borghini and Mazzieri (2017,2018)

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long history of incomplete published claims:
Lafontaine & Rozoy Actes du séminaire de théorie spectrale et géometrie (1999)

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NN3 & NN4 Invent. Mathematica 2022, retracted 2023

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Further static black holes with Λ?

O.J.C. Dias, G.W. Gibbons, J.E. Santos, B. Way, arXiv:2303.07361

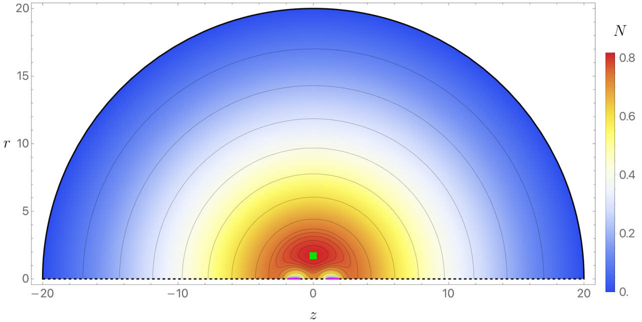

FIG. 3. Contour plot showing the level sets of the lapse function N. The cosmological horizon is the outer solid black semicircle. The horizon axes has the two black hole horizons as solid magenta lines, and the outer and inner axes in dashed black lines. The green square is where N takes its maximum value.

The near horizon geometry

key for understanding degenerate horizons

- Near a stationary *(event)* Killing horizon $\mathscr{H}$, in *Isenberg-Moncrief coordinates*, with $\partial_v g = 0$,

$$g = r\varphi dv^2 + 2dvdr + 2rh_a dx^a dv + h_{ab} dx^a dx^b \,,$$

$$\mathscr{H} = \{r = 0\}$$

- degenerate $\Longleftrightarrow \varphi|_{r=0} = 0$
- Moncrief ∼1970: in vacuum ∃ non-degenerate solutions with an arbitrary analytic h_{ab} *(no global regularity expected in general)*
- PTC, Reall, Tod 2006: Vacuum, *static*, degenerate, $\Lambda = 0$ $\Longrightarrow$ no solutions
- *Near horizon geometry*: set $\varphi(v, r, x^a) = rf(x^a)$ and

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Near-horizon equations

Near-horizon metric:

$$g = r^2 f dv^2 + 2dvdr + 2rX_a dx^a dv + g_{ab} dx^a dx^b \ ,$$

with $f = f(x^a)$, etc. Vacuum Einstein equations

$$\mathrm{Ric}(g) = \frac{1}{2} X^\flat \otimes X^\flat - \frac{1}{2} \mathcal{L}_X g + \lambda g \ , \qquad (0.1)$$

where $\mathrm{Ric}(g)$ is the Ricci tensor of g, $\mathcal{L}_X$ is the Lie derivative, the one–form $X^\flat$ is g–dual to X with respect to the metric g and λ is the cosmological constant.

(For physicists:

$$R_{ab} = \frac{1}{2} X_a X_b - \nabla_{(a} X_{b)} + \lambda g_{ab} \ .) \qquad (0.2)$$

Dunajski-Lucietti, 2306.17512: “The degenerate version” of Hawking’s rigidity theorem

Theorem

The extremal Kerr horizon (possibly with cosmological constant) is the unique solution to (0.1) *on* $M = S^2$.

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previous proofs assuming axisymmetry: Hajicek 1975; Pawlowski Lewandowski 2005; or assuming near-Kerr (PTC, Szybka, Tod 2018); or further global conditions (PTC 2023)

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previously: Hollands, Ishibashi (2015), under diophantine conditions

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Candidate Killing vector:

$$K_a := \Gamma X_a + (d\Gamma)_a \,, \quad \text{with } \Gamma \text{ so that } \nabla_a K^a = 0 \,. \tag{0.3}$$

The magic identity:

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Spacelike constraint equations

Initial data surface Σ, Riemannian metric g_{ij}, $i,j=1,\ldots,n$, symmetric tensor K_{ij} ("initial time derivative of the metric") the scalar constraint equation (Λ is the *cosmological constant*):

$$R(g_{ij}) = \cancel{16\pi T_{00}} + 2\Lambda + |K|^2 - (\mathrm{tr}K)^2\,,$$

and the *vector constraint equation*:

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Mao-Tao's version of the Corvino-Schoen gluing

Simplified proof; the original Corvino & Schoen theorem: to some nearby Kerr

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Mao, Tao, arXiv:2210.09437: can be done with optimal $1/r$ decay using a Green function for δR supported in a cone.

Theorem (Peter Hintz, arXiv:2210.13960)

Let (Σ, g, K) be a vacuum initial data set and suppose that there are no Killing vectors near $p \in \Sigma$. For every $\epsilon > 0$ sufficiently small there exists a vacuum initial data set which coincides with (g, K) outside an ϵ-neighborhood of p and coincides with a small Kerr black hole inside the neighborhood.

This can be done all over the place

Applications of spacelike gluing:

Gluing-in small black holes with $\Lambda = 0$; the Hintz black hole sprinkler (compare Anderson, Corvino, Pasqualotto arXiv:2301.08238)

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Theorem 1.1. *Let $N \in \mathbb{N}$. For $i = 1, \ldots, N$, fix points $p_i \in \partial M = \mathbb{S}^3 \subset \mathbb{R}^4$ and (subextremal) masses $0 < \mathfrak{m}_i < (3\Lambda)^{-1/2}$ such that the balance condition*

$$\sum_{i=1}^{N} \mathfrak{m}_i p_i = 0 \in \mathbb{R}^4. \tag{1.2}$$

holds. Then there exists a metric g solving the Einstein vacuum equation (1.1) in a neighborhood of ∂M with the following properties:

(1) *in a neighborhood of p_i, g is isometric to a Schwarzschild–de Sitter black hole metric with mass $\mathfrak{m}_i$, containing future affine complete event and cosmological horizons;*
(2) *outside a small neighborhood of $\{p_1, \ldots, p_N\}$, $\cos^2(s)g$ is smooth down to $s = \pi/2$, and asymptotic to the rescaled de Sitter metric $\cos^2(s)g_{\mathrm{dS}}$ at the rate $\cos^3(s)$.*

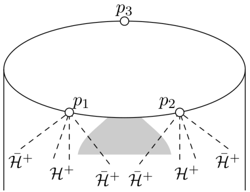

FIGURE 1.2. Illustration of Theorem 1.1. We glue SdS black holes into neighborhoods of the points p_i; only two black holes are shown here. The

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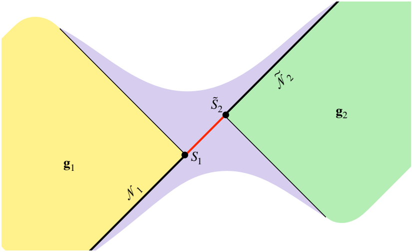

Figure: Gluing construction of Aretakis-Czimek-Rodnianski

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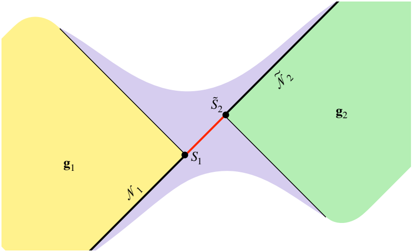

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ACR gluing

Characteristic gluing: implicit function theorem together with

Theorem (Aretakis, Czimek & Rodnianski, arXiv:2107.02449)

The C^2 linearised characteristic gluing at $(3+1)$-Minkowski is solvable up to a 10-dimensional space of obstructions.

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Theorem (PTC, Wan Cong and Finnian Gray, in preparation)

The C^k linearised characteristic gluing at $(n+1)$-Birmingham - Kottler is solvable up to a finite-dimensional space of obstructions.

$(n+1)$-Birmingham - Kottler: cross-section $\mathbf{S}$ compact Einstein spaces e.g. spheres, torus, higher genus; $\Lambda \in \mathbb{R}$, $m \in \mathbb{R}$.

Obstructions arise from kernels of linear elliptic operators on the cross-section **S** of the characteristic hypersurface; affected by *dimension* and *topology* of **S**, e.g.:

C^2-gluing with $m = 0$, $\Lambda = 0$	S^2	$\mathbb{T}^2$	S^4
dim. of obstruction space	10	7	30

Both a non-vanishing *mass m* and a non-zero *cosmological constant* Λ provide additional degrees of freedom to remove some of the obstructions, e.g.:

C^k-gluing	S^2, $m = 0$	S^2, $m = 0$	S^2, $m \neq 0$	**S**, $m \neq 0$
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CONJECTURE ("third law of black hole dynamics", Bardeen, Carter & Hawking (1973))

A black hole with zero surface-gravity cannot be formed in a dynamical process.

zero surface-gravity $\approx$ zero temperature

Theorem (Kehle & Unger, arXiv:2211.15742)

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Proof: use null gluing to an extreme Reissner-Nordström black hole.

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Proof: null gluing of a Minkowskian light-cone to a Kerr black hole

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ACR gluings: applications

"Obstruction-free gluing"

Theorem (Czimek & Rodnianski, arXiv:2210.09663)

Asymptotically flat initial data with mass m can be deformed, at large distances, to Kerr data *with any mass larger than m, same momentum, and with arbitrary remaining* asymptotic charges.

Remaining asymptotic charges: angular momentum and center of mass.

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Penrose inequality

3d, with optimal asymptotics: Benatti, Fogagnolo, Mazzieri, arXiv:2212.10215

Theorem

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$$c_p(\partial M)^{\frac{1}{3-p}} \le 2m \tag{0.4}$$

for any $1 < p \le 2$. Letting $p \to 1^+$ one obtains

$$\sqrt{\frac{|\partial M|}{16\pi}} \le m. \tag{0.5}$$

$$c_p(K) = \inf\left\{\frac{1}{4\pi}\left(\frac{p-1}{3-p}\right)^{p-1}\int_{M\setminus K}|Dv|^p \;\middle|\; v\in C^\infty_c(M),\ v \ge 1 \text{ on } K\right\}$$

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Theorem

Let (M, g) be a complete C^1_τ-asymptotically flat Riemannian 3-manifold, $\tau > 1/2$, with nonnegative scalar curvature and smooth, compact, minimal, connected and outermost boundary. Then,

$$c_p(\partial M)^{\frac{1}{3-p}} \leq 2m \qquad (0.4)$$

for any $1 < p \leq 2$. Letting $p \to 1^+$ one obtains

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq m. \qquad (0.5)$$

$$c_p(K) = \inf\left\{ \frac{1}{4\pi}\left(\frac{p-1}{3-p}\right)^{p-1} \int_{M\setminus K} |Dv|^p \;\middle|\; v \in C_c^\infty(M),\ v \geq 1 \text{ on } K \right\}.$$

The proof; u solves the p-Laplace equation

Agostiniani, Mantegazza, Mazzieri, Oronzio's version (arXiv:2205.11642) of an identity of Kijowski ($\sim$ 1982); see also Hirsch, Stern, Bray, Khuri, Kazaras 2102.11421, 1911.06754

$$\begin{aligned}
\operatorname{div} X = \frac{c_p^{\frac{p-1}{3-p}} |\nabla u|}{\left[\frac{3-p}{p-1}(1-u)\right]^{\frac{p-1}{3-p}+1}} \Bigg\{ & \underbrace{\frac{|\nabla u|^{p-1}}{c_p^{p-1}} - \frac{\mathrm{R}^{\Sigma}}{2}}_{\text{use Gauss-Bonnet on level sets of } u} \\
& + \frac{|\nabla^{\Sigma}|\nabla u||^2}{|\nabla u|^2} + \underbrace{\frac{R}{2}}_{\geq 0} + \frac{|\mathrm{h}|^2}{2} \\
& + \frac{5-p}{p-1}\left(\frac{|\nabla u|}{\frac{3-p}{p-1}(1-u)} - \frac{\mathrm{H}}{2}\right)^2 \Bigg\}
\end{aligned} \tag{0.6}$$

The vector field X

$$X = \frac{c_p^{\frac{p-1}{3-p}}}{\left[\frac{3-p}{p-1}(1-u)\right]^{\frac{p-1}{3-p}}}\left\{\frac{|\nabla u|^{p-2}\nabla u}{c_p^{p-1}} + \frac{\nabla|\nabla u| - \frac{\Delta u}{|\nabla u|}\nabla u}{\frac{3-p}{p-1}(1-u)} + \frac{|\nabla u|\nabla u}{\left[\frac{3-p}{p-1}(1-u)\right]^2}\right\}.$$