Casimir wormholes with a scalar field

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Plan of Talk

1. **Traversable Wormhole (TW)**
2. **Casimir Source + Scalar "Massless" Field**
3. **Casimir Source + Scalar "Massive" Field**

1. Traversable Wormhole: An Overview

- The Metric *(Morris & Thorne 1987)*:

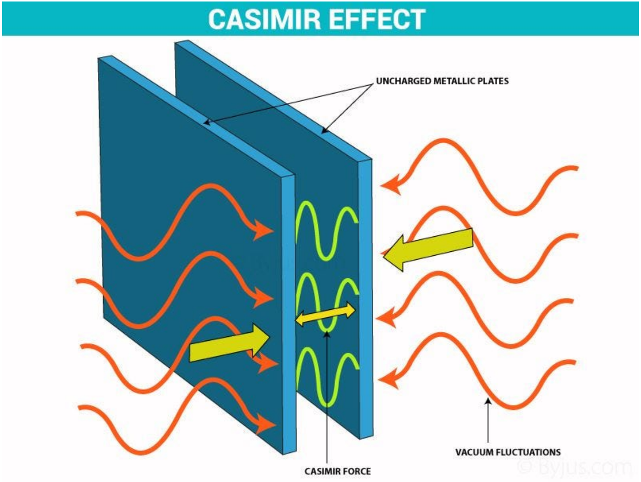
$$\mathrm{d}s^2 = -e^{2\Lambda(r)}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - b(r)/r} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\right)$$

$\Lambda(r) \rightarrow$ Redshift Function
$b(r) \rightarrow$ Shape Function

- Existence of a Throat r_0 $\rightarrow$ Flaring-out Condition: $\left.\frac{\mathrm{d}^2 r}{\mathrm{d}z^2}\right|_{r_0} > 0$

$\rightarrow$ Violation of the NEC: $\boxed{\rho(r_0) + p_r(r_0) \leq 0}$ Exotic Matter!!

(for $T^{\mu}_{\nu} = \mathrm{Diag}\left[-\rho(r), p_{\mathrm{r}}(r), p_{\mathrm{t}}(r), p_{\mathrm{t}}(r)\right]$)



$$\boxed{\rho_c = -\frac{\hbar c \pi^2}{720 d^4}}$$

Gravitational Action

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{1}{2}(\nabla\psi)^2 - V(\psi) + \mathcal{L}_{\mathrm{cas}} \right]$$

where $\kappa = 8\pi\ell_{\mathrm{P}}^2$ $(c = \hbar = 1)$ and $\psi = \psi(r)$

1. Einstein Field Equations (EFE):

$$\frac{b'(r)}{r^2} = \kappa\rho(r)$$

$$\frac{2\Lambda'(r)}{r}\left(1 - \frac{b(r)}{r}\right) - \frac{b(r)}{r^3} = \kappa p_{\mathrm{r}}(r)$$

$$\left(1 - \frac{b(r)}{r}\right)\left(\Lambda''(r) + \Lambda'(r)^2 + \frac{\Lambda'(r)}{r}\right) + \left(\frac{b(r) - rb'(r)}{2r^3}\right)\left(r\Lambda'(r) + 1\right)$$
$$= \kappa p_{\mathrm{t}}(r)$$

2. Scalar Field Equation:

$$\nabla^2\psi = \frac{\mathrm{d}V(\psi)}{\mathrm{d}\psi} = \frac{V'(r)}{\psi'(r)}$$

3. Stress-Energy Tensor (SET) Conservation:

$$\nabla_\mu T^\mu_\nu = 0 \quad \Rightarrow \quad p'_r(r) = \frac{2}{r}\left(p_t(r) - p_r(r)\right) - \left(\rho(r) + p_r(r)\right)\Lambda'(r)$$

where

$$\rho(r) = \frac{1}{2}\left(1 - \frac{b(r)}{r}\right)\psi'(r)^2 + V(r) + \rho_{\rm c}$$
$$p_{\rm r}(r) = \frac{1}{2}\left(1 - \frac{b(r)}{r}\right)\psi'(r)^2 - V(r) + p_{\rm r,c}$$
$$p_{\rm t}(r) = -\frac{1}{2}\left(1 - \frac{b(r)}{r}\right)\psi'(r)^2 - V(r) + p_{\rm t,c}$$

with

$$\rho_{\rm c} = -\frac{r_1^2}{\kappa d^4} = \frac{p_{\rm r,c}}{3} = -p_{\rm t,c}$$

and

$$r_1^2 = \frac{\hbar G}{c^3}\frac{\pi^3}{90} = \frac{\pi^3 \ell_{\rm P}^2}{90}$$

2. Set $V(\psi) = 0$

From SET Conservation:

$$\Lambda(r) = 2\ln\left(\frac{\bar{r}}{r}\right)$$

and

$$\rho\left(r\right) = \rho = \frac{C}{2\bar{r}^4} - \frac{r_1^2}{\kappa d^4} \gtreqless 0 \qquad \text{if} \qquad C \gtreqless 2\bar{r}^4 r_1^2 / \left(\kappa d^4\right)$$

Positive Energy Density ($r \in [r_0, \bar{r}]$)

$$C = \frac{2\bar{r}^2\left(2r_0^3 + 9r_0\bar{r}^2 - 8\bar{r}^3\right)}{3\kappa r_0^3}\,, \qquad b(r) = \frac{1}{r_0^2}\left(1 - \frac{8\bar{r}}{9r_0}\right)r^3 + \frac{8\bar{r}}{9}\,,$$

$$\bar{r} = \sqrt{\frac{2}{3}}\frac{d^2}{r_1}\,, \qquad r_0 < \bar{r} < 1.125 r_0$$

Need of an extra source:

$$T^{\mu}_{(\mathrm{E.S.})\nu} = \mathrm{Diag}\left[0, p_{1,\mathrm{r}}(r), p_{1,\mathrm{t}}(r), p_{1,\mathrm{t}}(r)\right]$$

with

$$p_{1,\mathrm{r}}(r) = \frac{4}{\kappa r^2} - \frac{8\bar{r}}{3\kappa r^3} - \frac{4}{3\kappa\bar{r}^2}$$
$$p_{1,\mathrm{t}}(r) = -\frac{4}{\kappa r^2} + \frac{4\bar{r}}{\kappa r^3}$$

Negative Energy Density ($r \in [r_0, \bar{r}]$)

$$C = \frac{2\bar{r}^2\left(2r_0^3 + 9r_0\bar{r}^2 - 8\bar{r}^3\right)}{3\kappa r_0^3}\,, \qquad b(r) = \frac{1}{r_0^2}\left(1 - \frac{8\bar{r}}{9r_0}\right)r^3 + \frac{8\bar{r}}{9}\,,$$

$$\bar{r} = \sqrt{\frac{2}{3}\frac{d^2}{r_1}}\,, \qquad 1.125r_0 < \bar{r} < 1.278r_0$$

Need of an extra source:

$$T^{\mu}_{(\mathrm{E.S.})\nu} = \mathrm{Diag}\left[0, p_{1,\mathrm{r}}(r), p_{1,\mathrm{t}}(r), p_{1,\mathrm{t}}(r)\right]$$

with

$$p_{1,\mathrm{r}}(r) = -\frac{4}{\kappa r^2} + \frac{8\bar{r}}{3\kappa r^3} + \frac{4}{3\kappa\bar{r}^2}$$
$$p_{1,\mathrm{t}}(r) = \frac{4}{\kappa r^2} - \frac{6\bar{r}}{\kappa r^3} + \frac{3\bar{r}^3 r_1^2}{\kappa d^4 r^3}$$

Vanishing Energy Density ($r \in [r_0, \bar{r}]$)

$$C = 2\bar{r}^4 r_1^2 / \left(\kappa d^4\right) \,, \qquad \rho(r) = p_{\rm t}(r) = 0 \,, \qquad p_{\rm r}(r) = -\frac{2r_1^2}{\kappa d^4} \,,$$

$$b(r) = r_0 \,, \qquad \bar{r} = \frac{9r_0}{8} \,, \qquad r_0 = \frac{8\sqrt{6}d^2}{27r_1}$$

Need of an extra source:

$$T^{\mu}_{({\rm E.S.})\nu} = {\rm Diag}\left[0, p_{1,{\rm r}}(r), p_{1,{\rm t}}(r), p_{1,{\rm t}}(r)\right]$$

with

$$p_{1,{\rm r}}(r) = \frac{4}{\kappa r^2} - \frac{3r_0}{\kappa r^3} - \frac{2r_1^2}{\kappa d^4},$$
$$p_{1,{\rm t}}(r) = -\frac{4}{\kappa r^2} + \frac{9r_0}{2\kappa r^3}$$

3. Set $V(\psi) \neq 0$ ($r \in [r_0, \bar{r}]$)

$$\Lambda(r) = 2\ln\left(\frac{\bar{r}}{r}\right), \qquad b(r) = \frac{4r}{3} - \frac{r^3}{3r_0^2}, \qquad V(r) = \frac{2}{3\kappa r^2} - \frac{r_1^2}{\kappa d^4},$$

$$\rho(r) = \frac{4}{3\kappa r^2} - \frac{1}{\kappa r_0^2}, \qquad \bar{r} = 2r_0$$

No extra source is needed!!

Thank you!