Quantum Euler angles and agency-dependent space-time

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Quantum space-time and symmetries

- Theoretical investigations in quantum gravity suggest that space-time itself should acquire «quantum features»
- Relativistic transformations also acquire quantum features, in order for the quantum space-time properties to be valid for every observer
- Observers are connected by quantum group transformations
- As a case study, we will consider the $SU_q(2)$ quantum group, to investigate purely rotated systems.

$SU(2)$ coordinatization and Euler Angles

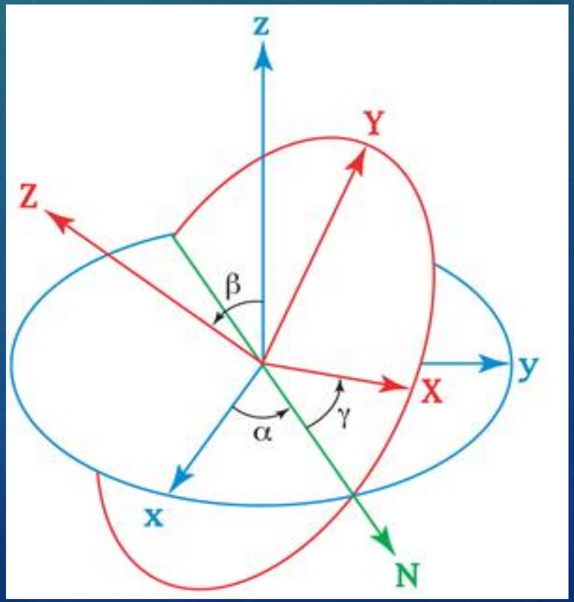
- In classical and quantum mechanics, rotation transformations are governed by the group SU(2)

$$SU(2) \ni U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \qquad a, c \in \mathbb{C} \ : \quad |a|^2 + |c|^2 = 1$$

$$a = e^{i\chi} \sin\left(\frac{\theta}{2}\right) \qquad c = e^{i\phi} \cos\left(\frac{\theta}{2}\right)$$

- SU(2) parameters and Euler Angles

$$\begin{cases} \theta = \beta \\ \chi = \frac{\alpha+\gamma}{2} \\ \phi = \frac{\pi}{2} - \frac{\alpha-\gamma}{2} \end{cases}$$



Link between $SU(2)$ and $SO(3)$

- The connection between SU(2) and classical rotations is established via the canonical homomorphism with SO(3).

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2) & \frac{i}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2) & a^*c + c^*a \\ \frac{i}{2}(a^2 + c^2 - (a^*)^2 - (c^*)^2) & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2) & -i\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - 2cc^* \end{pmatrix}$$

$\boldsymbol{SU_q(2)}$

- Parameters become the generators of $C_q\big(SU(2)\big)$, the algebra of complex functions on $SU(2)$

$$\begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \Rightarrow \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \qquad a, c \in C_q\big(SU(2)\big)$$

endowed with a non-commutative product realized by

$$ac = qca \qquad ac^* = qc^*a \qquad cc^* = c^*c$$

$$c^*c + a^*a = 1 \qquad aa^* - a^*a = (1 - q^2)c^*c$$

- q is a «small» deformation parameter, larger than 0 and close to 1.

Idempotent states on coquantum on Uq(2)Uq(2), SUq(2)SUq(2), and SOq(3) - Uwe Franz Adam Skalski and Reiji Tomatsu - Journal of Noncommutative Geometry

Homomorphism between $SU_q(2)$ and $SO_q(3)$

- $C_q(SO(3)) \coloneqq C_q(SU(2)/Z_2)$, realizing the q-analogue of the $SU(2)$ to $SO(3)$ homomorphism

- A 3x3 matrix representation is given by

$$R_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2) & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2) & \frac{1}{2}(1+q^2)(a^*c + c^*a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2) & -\frac{i}{2}(1+q^2)\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - (1+\mathrm{q}^2)cc^* \end{pmatrix}$$

- This is not a real valued matrix anymore, it contains operators

. Podles, “Symmetries of quantum spaces. subgroups and quotient spaces of quantumsu (2) andso (3) groups,” Communications in Mathematical Physics, vol. 170, no. 1, pp. 1–20, 1995

$SU_q(2)$ representations

- The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra is $H = H_\pi \oplus H_\rho$, where $H_\pi = L^2(S^1) \otimes L^2(S^1) \otimes \ell$ and $H_\rho = L^2(S^1)$

- $\rho(a)|\eta\rangle = e^{i\eta}|\eta\rangle$; $\rho(a^*)|\eta\rangle = e^{-i\eta}|\eta\rangle$; $\rho(c)|\eta\rangle = 0$; $\rho(c^*)|\eta\rangle = 0$;

- $\pi(a)|n,\delta,\epsilon\rangle = e^{i\epsilon}\sqrt{(1-q^{2n})}|n-1,\delta,\epsilon\rangle$; $\pi(a^*)|n,\delta,\epsilon\rangle = e^{-i\epsilon}\sqrt{(1-q^{2n+2})}|n+1,\delta,\epsilon\rangle$;

- $\pi(c)|n,\delta,\epsilon\rangle = e^{i\delta}q^n|n,\delta,\epsilon\rangle$; $\pi(c^*)|n,\delta,\epsilon\rangle = e^{-i\delta}q^n|n,\delta,\epsilon\rangle$;

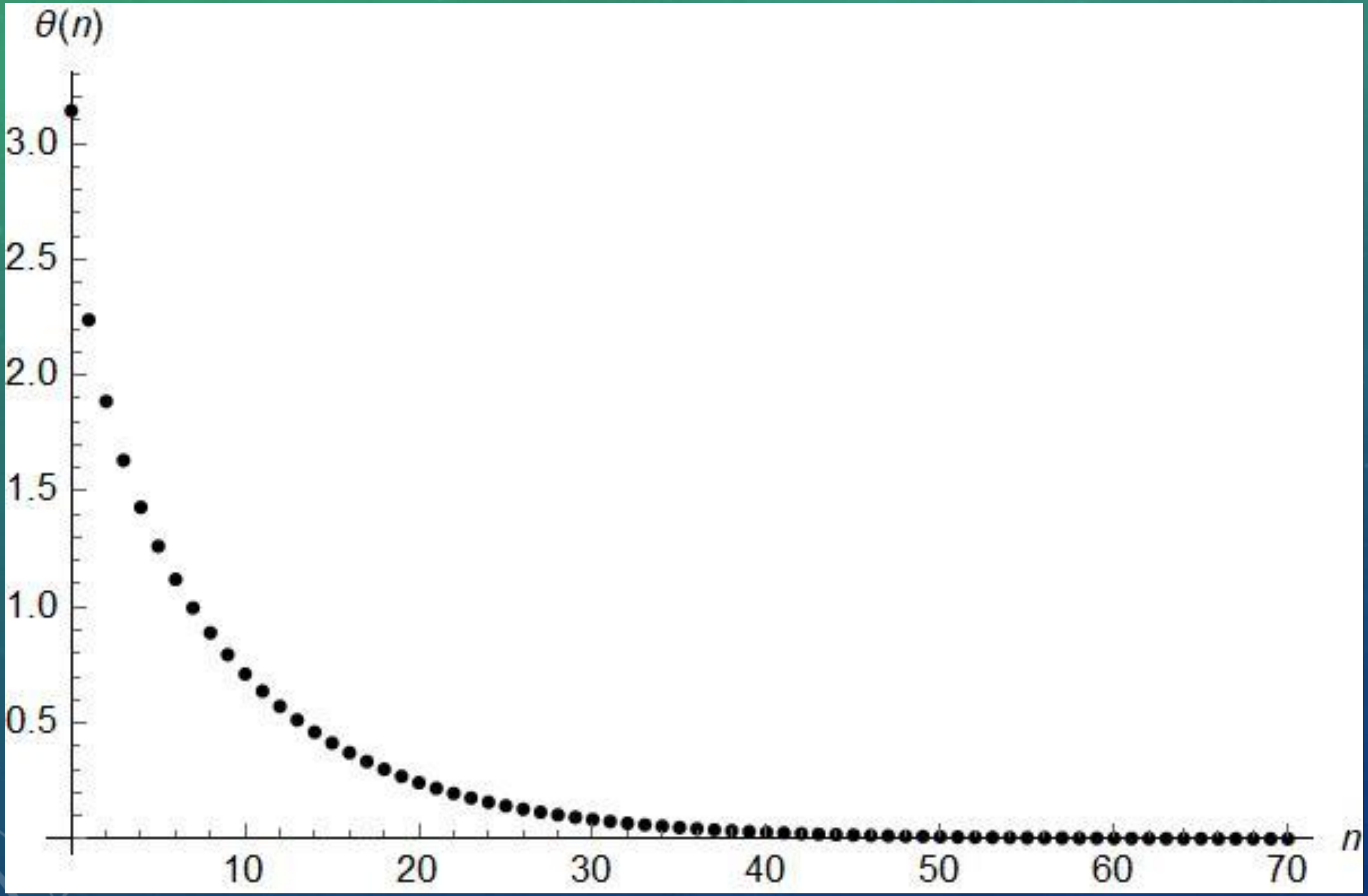
- $a = e^{i\chi}\cos\left(\frac{\theta}{2}\right)$ $c = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$ (Classical case)

Quantum Euler Angles (1)

- We promote the SU(2)-Euler Angles relations to the quantum case.
- Comparing the phases of a and c to their classical analogues, we identify ϵ with χ and δ with ϕ. They are continuous and play the same role as before.
- Exploiting the fact that c is a diagonal operator

$$q^n = Sin\left(\frac{\theta(n)}{2}\right) \leftrightarrow \theta(n) = 2Arcin(q^n)$$

Quantum Euler Angles (2)



$$\theta(n) = 2Arcin(q^n)$$

q=0.99

Physical interpretation and Quantum rotations

- A state $|\psi\rangle \in H$ is representative of the relative orientation between two reference frames, A and B.

- Our interpretation is that the mean value of R_q on $|\psi\rangle$ will give an estimate of the entries of the rotation matrix that connects A and B

$$\langle\psi|R_q|\psi\rangle_{\mathrm{ij}}$$

- However, due to non-commutatitvity, we will have a non vanishing variance for the matrix elements, in general:

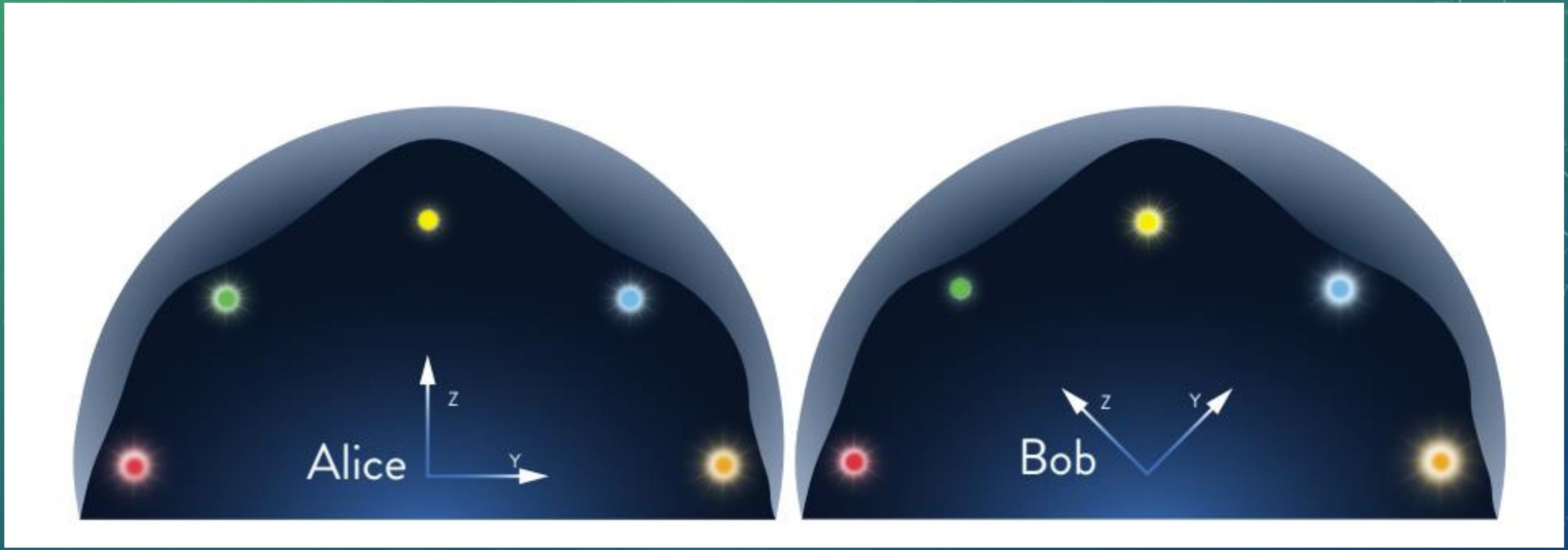
$$\Delta_{ij} = \sqrt{\langle\psi|R_q^2|\psi\rangle_{\mathrm{ij}} - \langle\psi|R_q|\psi\rangle_{\mathrm{ij}}^2}$$

Examples of Quantum rotations

- Basis states in representation ρ have $\Delta_{ij} = 0$ and the mean value of R_q on such states gives sharp rotations around the z-axis.

- Superpositions of basis states in representation π yield non-zero values of Δ_{ij} and the mean value of R_q gives rise to deformed rotation matrices, which reduce to standard rotation matrices about an axis in the $x - y$ plane in the commutative limit

Same stars, different skies



- Fuzziness of space-time points depends on the choices made by the observer. In this sense, the reconstructed space-time is agency-dependent.

Thanks for the attention!

Example: rotation around the z-axis

- Consider a state $|\chi\rangle$ in representation ρ. The mean value of the rotation matrix is:

$$\langle\chi|R_q|\chi\rangle_{\mathrm{ij}} = \begin{pmatrix} \cos(2\chi) & -\sin(2\chi) & 0 \\ \sin(2\chi) & \cos(2\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- It coincides with a standard $SO(3)$ rotation matrix. Indeed, computing the uncertainties, we have

$\Delta_{ij} = 0 \rightarrow$ Sharp rotations around the z-axis

«Physical» states construction

- To effectively describe rotations' deformations, we demand that our states of geometry $|\psi\rangle$ satisfy

$$\langle\psi|R_q|\psi\rangle_{ij} \to (R_{ij}) \qquad \Delta_{ij} \to 0 \qquad \text{when } q \to 1$$

where (R_{ij}) are the entries of a classical rotation matrix.

- Since (ϕ, χ) behave as in the classical case, we must look for states of the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n, \phi, \chi\rangle$$

heavily weighted around $\bar{n}$ and which satisfy the criteria above,
to properly describe a rotation deformation of Euler angles $(\phi, \chi, \theta(\bar{n}))$

Example: rotation of π around the x-axis

- Consider the state $|\psi\rangle = |n;\chi;\phi\rangle = \left|0;\frac{\pi}{2};0\right\rangle$. The relevant quantities, working at first order in $(1-q)$

$$\langle\psi|R_q|\psi\rangle = \begin{pmatrix} 1-(1-q) & 0 & 0 \\ 0 & -1+(1-q) & 0 \\ 0 & 0 & -1+2(1-q) \end{pmatrix} + \mathrm{o}(1-\mathrm{q})$$

$$\langle\psi|\Delta R_q|\psi\rangle = \begin{pmatrix} \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2(1-q)} & \sqrt{2(1-q)} & 0 \end{pmatrix} + o(1-q)$$

- As $q \to 1$, these correctly reproduce a rotation of π around the x-axis with null uncertainty.