Axially symmetric stationary gravitational perturbations of Kerr black hole

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1 Outline

1 Weyl – Lewis – Papapetrou vs Kerr

To study disk structures around black holes

- axial symmetry
- stationarity
- circularity

The Weyl – Lewis – Papapetrou metric reads

$$\mathbf{d}s^2 = e^{2\psi}\left(\mathbf{d}t - \Omega\mathbf{d}\varphi\right)^2 - e^{2(\gamma-\psi)}\left(\mathbf{d}z^2 + \mathbf{d}\varrho^2\right) - \varrho^2 e^{-2\psi}\mathbf{d}\varphi^2$$

where

$$\Delta_1\psi = -\frac{e^{4\psi}}{2\varrho^2}\,\nabla\Omega\cdot\nabla\Omega\,,\qquad \gamma_{,\varrho} = \varrho\left(\psi_{,z}^2 - \psi_{,\varrho}^2\right) - \frac{e^{4\psi}}{2\varrho^2}\,\left(\Omega_{,z}^2 - \Omega_{,\varrho}^2\right),$$

$$\Delta_{-1}\Omega = -4\nabla\psi\cdot\nabla\Omega\,,\qquad \gamma_{,z} = 2\varrho\psi_{,z}\psi_{,\varrho} - \frac{e^{4\psi}}{2\varrho^2}\,\Omega_{,z}\Omega_{,\varrho}$$

The first two equations can be reformulated in terms of *Ernst potentials*.

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Kerr metric – a special member of WLP class, in BL coordinates

$$\mathbf{d}s^2 = -\frac{\Delta}{\Sigma}\left(\mathbf{d}t - a\sin^2\theta\,\mathbf{d}\varphi\right)^2 + \frac{\Sigma}{\Delta}\,\mathbf{d}r^2 + \Sigma\,\mathbf{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\Big(\left(a^2 + r^2\right)\mathbf{d}\varphi - a\,\mathbf{d}t\Big)^2,$$

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We want to study the connection of

- direct linearization of WLP around Kerr background
- perturbative solutions obtained by Debye potential formalism

2 Outline

2 NP/GHP formalism

- particular realization of tetrad formalism
- equations for scalar quantities in a coordinate independent form
- particulary useful for Type D spacetimes

For example, the Maxwell tensor

$$\varphi_0 = \boldsymbol{F}_{ab} \boldsymbol{l}^a \boldsymbol{m}^b \,, \quad \varphi_1 = {}^1\!/_2\, \boldsymbol{F}_{ab} \left(\boldsymbol{l}^a \boldsymbol{n}^b + \boldsymbol{m}^a \bar{\boldsymbol{m}}^b \right) , \quad \varphi_2 = \boldsymbol{F}_{ab} \bar{\boldsymbol{m}}^a \boldsymbol{n}^b$$

and reconstruction

$$\boldsymbol{F} = -\boldsymbol{n} \wedge \bar{\boldsymbol{m}}\, \varphi_0 + (\boldsymbol{n} \wedge \boldsymbol{l} + \boldsymbol{m} \wedge \bar{\boldsymbol{m}})\, \varphi_1 - \boldsymbol{m} \wedge \boldsymbol{l}\, \varphi_2$$

2 Maxwell equations

$$\begin{aligned}
D\phi_1 - \bar{\delta}\phi_0 &= (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2\,,\\
D\phi_2 - \bar{\delta}\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2\,,\\
\Delta\phi_0 - \delta\phi_1 &= (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2\,,\\
\Delta\phi_1 - \delta\phi_2 &= \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2\,,
\end{aligned}$$

$$\begin{aligned}
\text{þ}\varphi_1 - \eth'\varphi_0 &= -\tau'\varphi_0 + 2\varrho\varphi_1 - \kappa\varphi_2\,,\\
\text{þ}\varphi_2 - \eth'\varphi_1 &= \sigma\varphi_0 - 2\tau'\varphi_1 + \varrho\varphi_2\,,
\end{aligned}$$

example

$$D\psi = \boldsymbol{l}^j\nabla_j\psi = \frac{1}{\sqrt{2}\,\Delta}\left[\left(r^2 + a^2\right)\partial_t + \Delta\,\partial_r + a\,\partial_\varphi\right]\psi\,,$$

Motivation:

- realistic situations $\longrightarrow$ numerical simulations
- simplification of the model $\longrightarrow$ (possibly) analytical solution

Simplifications?

- symmetries
- linearization (i.e. solving for test fields)
- algebraic type (D)

Advantages of this setup?

- contains black hole spacetimes
- Type D $\longrightarrow$ Teukolsky (decoupling of the MEq)
- Allows us to construct a Hertz potential in a special calibration (Debye potential) — one complex scalar function ψ
- $\psi \longrightarrow$ the whole field

Hertz potential

$$\boldsymbol{A}_a = \nabla^j \boldsymbol{P}_{aj} + \boldsymbol{G}_a$$

Hertz potential and gauge spinor in the principal dyad are chosen as follows

$$\begin{aligned} P_{AB} &= \chi_0\, \iota_A \iota_B - \chi_1 \left(o_A \iota_B + o_B \iota_A \right) + \chi_2\, o_A o_B \,, \\ G_{AA'} &= \quad 2 \left(\bar{\tau}' \bar{\chi}_0 - \bar{\varrho} \bar{\chi}_1 \right) \iota_A \bar{\iota}_{A'} - 2 \left(\bar{\varrho}' \bar{\chi}_0 - \bar{\tau} \bar{\chi}_1 \right) o_A \bar{\iota}_{A'} \\ &\quad - 2 \left(\bar{\tau}' \bar{\chi}_1 - \bar{\varrho} \bar{\chi}_2 \right) \iota_A \bar{o}_{A'} + 2 \left(\bar{\varrho}' \bar{\chi}_1 - 2\bar{\tau} \bar{\chi}_2 \right) o_A \bar{o}_{A'} \,, \end{aligned}$$

and

$$\varphi_0 = þþ\bar{\chi}_0 \,, \qquad \varphi_1 = 2(þð' + \tau' þ)\bar{\chi}_0 \,, \qquad \varphi_2 = ð'ð'\bar{\chi}_0 \,.$$

3 Master Equations

Teukolsky Master Equation

$$\left[(\text{þ} - \bar{\varrho})\left(\text{þ}' + \varrho'\right) - (\eth - \bar{\tau}')\,(\eth' + \tau')\right] \psi_2^{-2/3}\varphi_0 = J_0\,,$$

sources encoded in (Green function – not physical sources)

$$J_0 = (\eth - \bar{\tau}')\,\psi_2^{-2/3} J_l - (\text{þ} - \bar{\varrho})\,\psi_2^{-2/3} J_m\,.$$

$$\varphi_0 = \int_0^\infty \int_0^\pi G(r,\theta,r',\theta') J_0(r',\theta',r_0,\theta_0)\Sigma(r',\theta') \sin\theta'\,\mathrm{d}r'\mathrm{d}\theta'\,.$$

The Deybe potential equation (Andersson, Backdähl, Blue – fundamental spinor operators)

$$\left[\left(\text{þ}' - \varrho'\right)(\text{þ} + \bar{\varrho}) - (\eth - \tau)\,(\eth' + \bar{\tau})\right] \bar{\psi} = 0\,.$$

In the BL coordinates these Eqs. are separable. Standard way of solving: series expansion (quite general)

3 Axial symmetry + stationarity

Stationarity + Axiall symmetry

$$\text{TME, DEq} \implies \text{Laplace Eq (in 2s+3 dim)}$$

Green function for TME

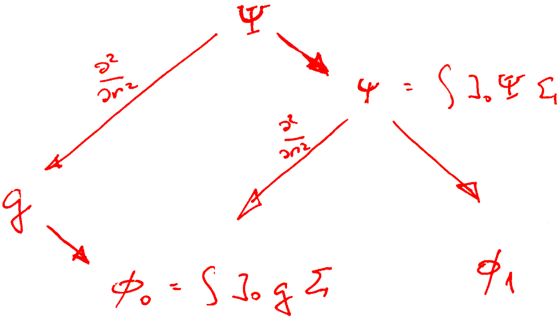
$$g = \frac{d(z_0, -\varrho_0)}{\varrho^2}\left[-E(m) + \left(1 + \frac{2\varrho_0\varrho}{d(z_0, -\varrho_0)^2}\right) K(m)\right] .$$

Physical four-current

$$\boldsymbol{J} = j_0 \boldsymbol{\partial_t} + j_3 \boldsymbol{\partial_\varphi} \, ,$$

Simplified expression for the sources of TME reads as follows

$$J_0 = \frac{1}{2\rho\Sigma}\left[-\frac{\partial}{\partial\theta}\left(\rho^2 j_0\right) + ia\sin\theta\frac{\partial}{\partial r}\left(\rho^2 j_0\right) + \frac{\partial}{\partial\theta}\left(a\sin^2\theta\,\rho^2 j_3\right) - i\frac{\partial}{\partial r}\left(\left(a^2 + r^2\right)\sin\theta\,\rho^2 j_3\right)\right]$$


Ψ
∂²/∂r²
ψ = ∫ J₀ Ψ Σ₁
∂²/∂r²
g
φ₀ = ∫ J₀ g Σ
φ₁

3 Superpotential integration?

How to integrate the superpotetnial?

- direct attempt – impossible

- transform to Weyl coordinates (Laplace)
- integrate on the axis (the solution is given by its values on the axis)
- integrate away from the axis

$$\Xi_{\mathbf{r}}(0, z) = \frac{\pi}{\sin\theta_0} \frac{\sqrt{(z - z_0)^2 + \varrho_0^2}}{z^2 - \beta^2} \,.$$

$$\Xi_{\mathbf{r}}(\varrho, z) = \frac{2}{\pi} \int_0^{\pi} \Xi_{\mathbf{r}}\left(0, z + i\varrho\cos\alpha\right) \sin^2\alpha \, \mathrm{d}\alpha = \frac{2}{\sin\theta_0} \int_0^{\pi} \frac{\sqrt{\varrho_0^2 + \left(z + i\varrho\cos\alpha - z_0\right)^2}}{(z + i\varrho\cos\alpha)^2 - \beta^2} \sin^2\alpha \, \mathrm{d}\alpha \,.$$

3 Intermezzo: Elliptic integrals

$$E(m) = \int_0^{1/2\,\pi} \sqrt{1 - m \sin^2\theta}\, \mathrm{d}\theta\,,$$

$$K(m) = \int_0^{1/2\,\pi} \frac{1}{\sqrt{1 - m \sin^2\theta}}\, \mathrm{d}\theta\,,$$

$$\Pi(n|m) = \int_0^{1/2\,\pi} \frac{1}{\left(1 - n \sin^2\theta\right)\sqrt{1 - m \sin^2\theta}}\, \mathrm{d}\theta\,.$$

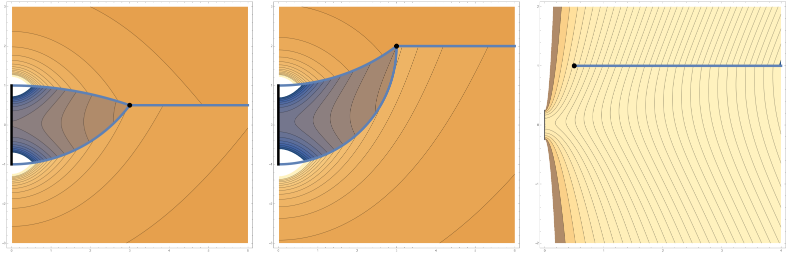
Branch cut $n \in \langle 1, \infty \rangle$

$$\lim_{\epsilon \to +0} \Pi(n - i\epsilon|m) = \Pi(n|m)\,,$$

$$\lim_{\epsilon \to +0} \Pi(n + i\epsilon|m) = \Pi(n|m) + \frac{\pi}{\sqrt{1-n}\sqrt{1-\frac{m}{n}}}\,.$$

3 Superpotential

$$\Xi_{\mathbf{r}} = \frac{1}{\varrho^2 d(z_0,\varrho_0)}\Bigg[-id(z_0,\varrho_0)^2 E(\mu') + 2\varrho_0\left(4z - h(z_0,\varrho_0)\right)K(\mu') - 4(z+z_0)\varrho_0\Pi\left(\frac{h(z_0,-\varrho_0)}{h(z_0,\varrho_0)}\,\middle|\,\mu'\right)$$
$$+\frac{2\varrho_0 d(\beta,0)^2}{\beta}\Pi\left(\frac{(z_0-\beta-i\varrho_0)\bar{h}(z_0,-\varrho_0)}{(z_0-\beta+i\varrho_0)\bar{h}(z_0,\varrho_0)}\,\middle|\,\mu'\right) - \frac{2\varrho_0 d(\beta,0)^2}{\beta}\Pi\left(\frac{(z_0+\beta-i\varrho_0)\bar{h}(z_0,-\varrho_0)}{(z_0+\beta+i\varrho_0)\bar{h}(z_0,\varrho_0)}\,\middle|\,\mu'\right)\Bigg] + c.c.\,.$$

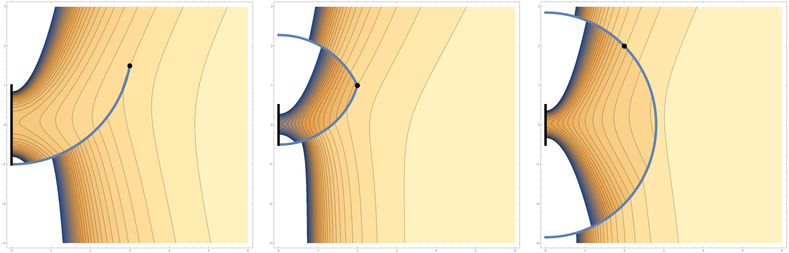


(a) (b) (c)

Figure: The contourplot of the Debye superpotential $\Xi_{\mathbf{r}}$ in the Weyl coordinates (ϱ, z).

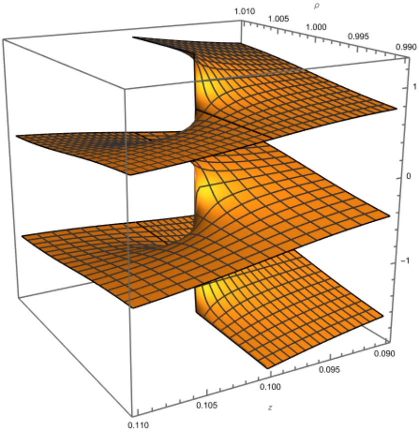
3 Superpotential

$$\Xi_{\mathbf{r}} = \frac{1}{\varrho^2 d(z_0, \varrho_0)} \Bigg[-i d(z_0, \varrho_0)^2 E(\mu') + 2\varrho_0 \left(4z - h(z_0, \varrho_0)\right) K(\mu') - 4(z+z_0)\varrho_0 \Pi\left(\frac{h(z_0, -\varrho_0)}{h(z_0, \varrho_0)} \middle| \mu'\right)$$
$$+ \frac{2\varrho_0 d(\beta, 0)^2}{\beta} \Pi\left(\frac{(z_0 - \beta - i\varrho_0)\bar{h}(z_0, -\varrho_0)}{(z_0 - \beta + i\varrho_0)\bar{h}(z_0, \varrho_0)} \middle| \mu'\right) - \frac{2\varrho_0 d(\beta, 0)^2}{\beta} \Pi\left(\frac{(z_0 + \beta - i\varrho_0)\bar{h}(z_0, -\varrho_0)}{(z_0 + \beta + i\varrho_0)\bar{h}(z_0, \varrho_0)} \middle| \mu'\right) \Bigg] + c.c.\,.$$



(a) $\Xi_{\mathbf{0}}$ (b) $\Xi_{\mathbf{1}}$ (c) $\Xi_{\mathbf{2}}$

Figure: The contourplot of the Debye superpotential $\Xi_{\mathbf{r}}$ in the Weyl coordinates (ϱ, z).



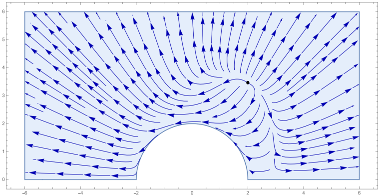
(a)

Uniformly charged ring

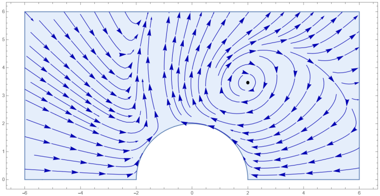
$$j_0 = \hat{j}_0(r_0,\theta_0)\,\frac{\delta\,(r-r_0)}{\Sigma(r_0,\theta_0)}\,\frac{\delta\,(\theta-\theta_0)}{\sin\theta_0}\,, \qquad\qquad j_3 = 0\,.$$

Evaluating the integral leads to the Debye potential of the charged ring

$$\bar{\psi}_{\mathbf{ring}} = \frac{\hat{j}_0(r_0,\theta_0)}{2\overline{\rho(r_0,\theta_0)}}\left(\cot\theta_0 + \frac{\partial}{\partial\theta_0} - ia\sin\theta_0\,\frac{\partial}{\partial r_0}\right)\Psi\,.$$



(b) Integral curves of ***E*** (ring).



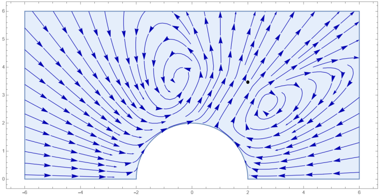
(c) Integral curves of ***B*** (ring).

Circular current

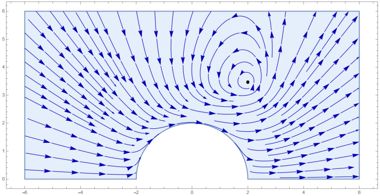
$$j_0 = 0\,, \qquad j_3 = \hat{j}_3(r_0,\theta_0)\frac{\delta\left(r-r_0\right)}{\Sigma(r_0,\theta_0)}\frac{\delta\left(\theta-\theta_0\right)}{\sin\theta_0}\,.$$

Evaluating the integral leads to the Debye potential of the current loop

$$\bar{\psi}_{\mathbf{current}} = \hat{j}_3(r_0,\theta_0)\frac{\sin\theta_0}{2\overline{\rho(r_0,\theta_0)}}\left(-ir_0 - a\sin\theta_0\frac{\partial}{\partial\theta_0} + i\left(r_0^2+a^2\right)\frac{\partial}{\partial r_0}\right)\Psi\,.$$



(d) Integral curves of ***E*** (circular current).



(e) Integral curves of ***B*** (circular current).

4 Outline

4 TME, s=2

Teukolsky Master equation

$$\Big[(\text{þ} - \bar{\varrho} - 4\varrho)\left(\text{þ}' - \varrho'\right) - (\eth - \bar{\tau}' - 4\tau)\left(\eth' - \tau'\right) - 3\psi_2\Big]\dot{\psi}_0 = T_0\,,$$

sources

$$\begin{aligned} T_0 = \;& (\eth - 4\tau - \bar{\tau}')\left[(\text{þ} - 2\bar{\varrho})\, T_{\boldsymbol{lm}} - (\eth - \bar{\tau}')\, T_{\boldsymbol{ll}}\right] \\ &+ (\text{þ} - 4\varrho - \bar{\varrho})\left[(\eth - 2\bar{\tau}')\, T_{\boldsymbol{lm}} - (\text{þ} - \bar{\varrho})\, T_{\boldsymbol{mm}}\right] \end{aligned}$$

Given T_0, we

1. solve for $\dot{\psi}_0$ and then
2. evaluate $\dot{\psi}_0$ on the axis
3. solve for the Debye potential on the axis
4. evaluate the integral to get the potential everywhere

4 Debey potentials for gravitational perturbations - ORG

$$\boldsymbol{h}_{ab}^{(out)} = \left(X_o + \bar{X}_o\right) \boldsymbol{l}_a \boldsymbol{l}_b + \bar{Y}_o \boldsymbol{m}_a \boldsymbol{m}_b + Y_o \bar{\boldsymbol{m}}_a \bar{\boldsymbol{m}}_b - 2\bar{Z}_o \boldsymbol{l}_{(a} \boldsymbol{m}_{b)} - 2Z_o \boldsymbol{l}_{(a} \bar{\boldsymbol{m}}_{b)}$$

with

$$\begin{aligned} X_o &= (\eth\eth + 2\tau'\eth)\,\chi_{[-4,0]}\,, \\ Y_o &= (\text{þþ} + 2\varrho\text{þ})\,\bar{\chi}_{[0,-4]}\,, \\ Z_o &= (\text{þ}\eth' + (\tau + \tau')\,\text{þ} + \varrho\eth)\,\bar{\chi}_{[0,-4]} \end{aligned}$$

and

$$\begin{aligned} 2\dot{\psi}_0 &= \text{þþþþ}\,\bar{\hat{\psi}}_{[0,-4]}\,, \\ 2\dot{\psi}_4 &= \eth'\eth'\eth'\eth'\,\bar{\hat{\psi}}_{[0,-4]} - 3\psi_2\left[\tau'\eth - \tau\eth' + \varrho\text{þ}' - \varrho'\text{þ} - 2\psi_2\right]\hat{\psi}_{[-4,0]}\,, \end{aligned}$$

Out[337]= $\bar{m}_a\ \bar{m}_b\ Y_\circ - l_a\ \bar{m}_b\ Z_\circ + l_b\left(l_a\left(X_\circ + \bar{X}_\circ\right) - \bar{m}_a\ Z_\circ\right) - l_b\ m_a\ \bar{Z}_\circ - \frac{1}{4}\ m_b\left(-4\ m_a\ \bar{Y}_\circ + 4\ l_a\ \bar{Z}_\circ\right)$

Out[338]= $l_a\ l_b\ X_\circ + l_a\ l_b\ \bar{X}_\circ + \bar{m}_a\ \bar{m}_b\ Y_\circ + m_a\ m_b\ \bar{Y}_\circ - l_b\ \bar{m}_a\ Z_\circ - l_a\ \bar{m}_b\ Z_\circ - l_b\ m_a\ \bar{Z}_\circ - l_a\ m_b\ \bar{Z}_\circ$

Out[339]//TableForm=

$\frac{1}{2}\left(x + \bar{x} + 2\,i\,a\,(z - \bar{z})\,\mathrm{Sin}[\theta] - a^2\,(y + \bar{y})\,\mathrm{Sin}[\theta]^2\right)$	$-\frac{\left(x-\bar{x}-i\,a\,(z-\bar{z})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]}{2\,\Delta[r]}$	$\frac{1}{2}\left(z + \bar{z} + i\,a\,(y - \bar{y})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]$	$\frac{1}{2}\,\mathrm{Sin}[\theta]\left(-i\,(z - \bar{z})\left(a^2 + r^2\right) + a\left(-x - \bar{x}\right.\right.$
$-\frac{\left(x+\bar{x}+i\,a\,(z-\bar{z})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]}{2\,\Delta[r]}$	$\frac{(x+\bar{x})\,\Sigma[r,\theta]^2}{2\,\Delta[r]^2}$	$-\frac{(z+\bar{z})\,\Sigma[r,\theta]^2}{2\,\Delta[r]}$	$\frac{\mathrm{Sin}[\theta]\left(i\,(z-\bar{z})\left(a^2+r^2\right)+a\,(x+\bar{x})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]}{2\,\Delta[r]}$
$\frac{1}{2}\left(z + \bar{z} + i\,a\,(y - \bar{y})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]$	$-\frac{(z+\bar{z})\,\Sigma[r,\theta]^2}{2\,\Delta[r]}$	$\frac{1}{2}\,(y + \bar{y})\,\rho cc[r,\theta]^2\,\rho nc[r,\theta]^2$	$-\frac{1}{2}\,i\,\mathrm{Sin}[\theta]\left(\left(y - \bar{y}\right)\left(a^2 + r^2\right) - i\,a\,(z + \bar{z}\right.$
$\frac{1}{2}\,\mathrm{Sin}[\theta]\left(-i\,(z - \bar{z})\left(a^2 + r^2\right) + a\left(-x - \bar{x} + (y + \bar{y})\left(a^2 + r^2\right)\right)\mathrm{Sin}[\theta] - i\,a^2\,(z - \bar{z})\,\mathrm{Sin}[\theta]^2\right)$	$\frac{\mathrm{Sin}[\theta]\left(i\,(z-\bar{z})\left(a^2-r^2\right)-a\,(x-\bar{x})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]}{2\,\Delta[r]}$	$-\frac{1}{2}\,i\,\mathrm{Sin}[\theta]\left(\left(y - \bar{y}\right)\left(a^2 + r^2\right) - i\,a\,(z + \bar{z})\,\mathrm{Sin}[\theta]\right)\Sigma[r,\theta]$	$\frac{1}{2}\,\mathrm{Sin}[\theta]^2\left(-\left((y + \bar{y})\left(a^2 + r^2\right)^2\right) + 2\,i\,a\,(z\right.$

In[309]:= sXYZ

Out[309]= $\left\{X_\circ \to x,\ \bar{X}_\circ \to \bar{x},\ Y_\circ \to y\,\rho nc[r,\theta]^2,\ \bar{Y}_\circ \to \bar{y}\,\rho cc[r,\theta]^2,\ Z_\circ \to z\,\rho nc[r,\theta],\ \bar{Z}_\circ \to \bar{z}\,\rho cc[r,\theta]\right\}$

Let us use

$$\boldsymbol{h}_{ab} = \boldsymbol{h}_{ab}^{(in)} + \boldsymbol{h}_{ab}^{(out)}$$

Tedious work leads to the following expressions in BL coordinates (t, r, θ, φ)

$$\boldsymbol{h}_{ab} = \begin{pmatrix} h_{tt} & 0 & 0 & h_{t\varphi} \\ 0 & h_{rr} & h_{r\theta} & 0 \\ 0 & h_{r\theta} & h_{\theta\theta} & 0 \\ h_{t\varphi} & 0 & 0 & h_{\varphi\varphi} \end{pmatrix}$$

with

$$\begin{aligned} \boldsymbol{h} = & (x + \bar{x}) \left[\left(\mathbf{d}t - a \sin^2 \theta \, \mathbf{d}\varphi \right)^2 + \frac{\Sigma}{\Delta} \, \mathbf{d}r^2 \right] \\ & - (y + \bar{y}) \left[\left(a\mathbf{d}t - (r^2 + a^2)\mathbf{d}\varphi \right)^2 \sin^2 \theta - \Sigma^2 \mathbf{d}\theta^2 \right] \\ & - 2i(z - \bar{z}) \sin \theta \left[-a\mathbf{d}t + (r^2 + a^2)\mathbf{d}\varphi \right] \left(\mathbf{d}t - a \sin^2 \theta \mathbf{d}\varphi \right) \\ & - 2(z + \bar{z}) \frac{\Sigma^2}{\Delta} \, \mathbf{d}r \, \mathbf{d}\theta \end{aligned}$$

4 Weyl – Lewis – Papapetrou

WLP metric reads

$$\mathrm{d}s^2 = e^{2\psi}\left(\mathrm{d}t - \Omega\mathrm{d}\varphi\right)^2 - e^{2(\gamma-\psi)}\left(\mathrm{d}z^2 + \mathrm{d}\varrho^2\right) - \varrho^2 e^{-2\psi}\mathrm{d}\varphi^2$$

and linearize around known background

$$\psi \to \psi_0 + \epsilon\psi_1\,, \qquad \lambda \to \lambda_0 + \epsilon\lambda_1\,, \qquad \Omega \to \Omega_0 + \epsilon\Omega_1\,.$$

We may as well consider

$$\mathrm{d}s^2 = e^{2\psi}\left(\mathrm{d}t - \Omega\mathrm{d}\varphi\right)^2 - e^{-2\psi}\left(\gamma_{MN}\mathrm{d}x^M\mathrm{d}x^N + \varrho^2\mathrm{d}\varphi^2\right)$$

in BL coordinates we get

$$\boldsymbol{h}_{ab} = \begin{pmatrix} h_{tt} & 0 & 0 & h_{t\varphi} \\ 0 & h_{rr} & 0 & 0 \\ 0 & 0 & h_{\theta\theta} & 0 \\ h_{t\varphi} & 0 & 0 & h_{\varphi\varphi} \end{pmatrix} \tag{1}$$

$$\begin{aligned}
\boldsymbol{h}_{tt} &= 2\left(\Delta - a^2 \sin^2\theta\right)\Sigma^{-1}\psi_1\,, \\
\boldsymbol{h}_{t\varphi} &= 2aMr\sin^2\theta\Sigma^{-1}\psi_1 + (a^2\sin^2\theta - \Delta)\Sigma^{-1}\Omega_1\,, \\
\boldsymbol{h}_{\varphi\varphi} &= -\frac{\varrho^2\Sigma^2 + 4a^2M^2r^2\sin^4\theta}{\Sigma\left(a^2\sin^2\theta - \Delta\right)}\psi_1 - 4aMr\sin^2\theta\Sigma^{-1}\Omega_1\,, \\
\boldsymbol{h}_{rr} &= 2\Sigma\Delta^{-1}\left(\psi_1 - \gamma_1\right)\,, \\
\boldsymbol{h}_{\theta\theta} &= 2\Sigma\left(\psi_1 - \gamma_1\right)\,.
\end{aligned}$$

$$\boldsymbol{h}_{WLP} = \boldsymbol{h} + \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}} \boldsymbol{g}_{Kerr}$$

for calibration vector

$$\boldsymbol{\xi} = f_r(r,\theta)\boldsymbol{\partial}_r + f_\theta(r,\theta)\boldsymbol{\partial}_\theta$$

we have to solve

$$-\frac{\Sigma^2}{\Delta}(z+\bar{z}) = \frac{\Sigma}{2}\left(\frac{\partial}{\partial r}\frac{f_r}{\Sigma} + \frac{\partial}{\partial \theta}\frac{f_\theta}{\Sigma}\right)$$

and we have an algebraic constraint

$$f_\theta + \frac{\Delta'}{\Delta}\tan\theta\, f_r = \Sigma\tan\theta\left[(y+\bar{y}) + \frac{1}{\Delta}(x+\bar{x})\right]$$

4 Dragging

On Schwarzschild background we get

$$\Omega_1 = i\,\frac{r^4 \sin\theta}{\Delta}\,(z - \bar{z})$$

which solves linearized WLP eq

$$\Delta_{-1}\Omega_1 + 4\nabla\psi_0 \cdot \nabla\Omega_1 = 0$$

4 Full set of Equations

```
(* full set of equation, i.e. h[χ] == h[WLP] + Lie[ξ][Kerr] *)
ceq // ShowEq
```

1 $\mathrm{Sin}[\theta]^2\left(-\left(a^2+r^2\right)^2(y+y)+a\,\mathrm{Sin}[\theta]\left(2\,i\left(a^2+r^2\right)(z-z)+a\,(x+x)\,\mathrm{Sin}[\theta]\right)\right) == \frac{2\,\mathrm{Sin}[\theta]^2\left(\left(4a^2M^2r^2\,\mathrm{Sin}[\theta]^2+\Delta[r]\,\Sigma[r,\theta]^2\right)\psi1[\rho,z]-2\,a\,M\,r\,\Delta s[r,\theta]\,\Omega1[\rho,z]\right)}{\Delta s[r,\theta]\,\Sigma[r,\theta]}+\frac{fr[r,\theta]\,\mathrm{Sin}[\theta]^4\,\Upsilon[r,\theta]^2\,\partial_r\frac{\mathrm{Csc}[\theta]^2\,\Sigma[r,\theta]}{\Upsilon[r,\theta]}}{2\,\Sigma[r,\theta]^2}+\frac{f\theta[r,\theta]\,\mathrm{Sin}[\theta]^4\,\Upsilon[r,\theta]^2\,\partial_\theta\frac{\mathrm{Csc}[\theta]^2\,\Sigma[r,\theta]}{\Upsilon[r,\theta]}}{2\,\Sigma[r,\theta]^2}$

2 $(x+\bar{x})+a\,\mathrm{Sin}[\theta]\,(2\,i\,(z-\bar{z})-a\,(y+\bar{y})\,\mathrm{Sin}[\theta]) == -\frac{a^2\,M\,r\,f\theta[r,\theta]\,\mathrm{Sin}[2\theta]}{\Sigma[r,\theta]^2}+\frac{M\,fr[r,\theta]\,\Sigma w[r,\theta]}{\Sigma[r,\theta]^2}+\frac{2\,\Delta s[r,\theta]\,\psi1[\rho,z]}{\Sigma[r,\theta]}$

3 $(y+\bar{y})\,\Sigma[r,\theta]^2 == -2\,\Sigma[r,\theta]\,(\gamma1[\rho,z]-\psi1[\rho,z])-\Sigma[r,\theta]\,f\theta^{(0,1)}[r,\theta]-\frac{1}{2}f\theta[r,\theta]\,\Sigma^{(0,1)}[r,\theta]-\frac{1}{2}fr[r,\theta]\,\Sigma^{(1,0)}[r,\theta]$

4 $\frac{(x-\bar{x})\,\Sigma[r,\theta]^2}{\Delta[r]^2} == -\frac{2\,\Sigma[r,\theta]\,(\gamma1[\rho,z]-\psi1[\rho,z])}{\Delta[r]}-\frac{1}{2}fr[r,\theta]\,\mathrm{Sin}[\theta]^2\,\partial_r\frac{\mathrm{Csc}[\theta]^2\,\Sigma[r,\theta]}{\Delta[r]}-\frac{f\theta[r,\theta]\,\Sigma^{(0,1)}[r,\theta]}{2\,\Delta[r]}-\frac{\Sigma[r,\theta]\,fr^{(1,0)}[r,\theta]}{\Delta[r]}$

5 $-\frac{(z-\bar{z})\,\Sigma[r,\theta]^2}{\Delta[r]} == -\frac{\Sigma[r,\theta]\,fr^{(0,1)}[r,\theta]}{2\,\Delta[r]}-\frac{1}{2}\Sigma[r,\theta]\,f\theta^{(1,0)}[r,\theta]$

6 $\mathrm{Sin}[\theta]\left(a\left(-(x+\bar{x})+\left(a^2+r^2\right)(y+\bar{y})\right)\mathrm{Sin}[\theta]-i\,(z-\bar{z})\left(2a^2+\Sigma s[r,\theta]\right)\right) == \frac{4\,a\,M\,r\,\mathrm{Sin}[\theta]^2\,\psi1[\rho,z]-\Delta s[r,\theta]\,\Omega1[\rho,z]}{\Sigma[r,\theta]}-\frac{a\,M\,r^2\,fr[r,\theta]\,\mathrm{Sin}[\theta]^4\,\partial_r\frac{\mathrm{Csc}[\theta]^2\,\Sigma[r,\theta]}{r}}{\Sigma[r,\theta]^2}-\frac{a\,M\,r^2\,f\theta[r,\theta]\,\mathrm{Sin}[\theta]^4\,\partial_\theta\frac{\mathrm{Csc}[\theta]^2\,\Sigma[r,\theta]}{r}}{\Sigma[r,\theta]^2}$

$$\Big\{\psi1[\rho,z] \to \frac{\Sigma[r,\theta]}{2\,\Delta s[r,\theta]}\left((x+\bar{x})+2\,i\,a\,(z-\bar{z})\,\mathrm{Sin}[\theta]-a^2\,(y+\bar{y})\,\mathrm{Sin}[\theta]^2\right)+\frac{M}{2\,\Delta s[r,\theta]\,\Sigma[r,\theta]}\left(a^2\,r\,f\theta[r,\theta]\,\mathrm{Sin}[2\theta]-fr[r,\theta]\,\Sigma w[r,\theta]\right),$$

$$\gamma\psi[\rho,z] \to -\frac{1}{2}(y+\bar{y})\,\Sigma[r,\theta]-\frac{1}{2}f\theta^{(0,1)}[r,\theta]-\frac{1}{4\,\Sigma[r,\theta]}\left(f\theta[r,\theta]\,\Sigma^{(0,1)}[r,\theta]+fr[r,\theta]\,\Sigma^{(1,0)}[r,\theta]\right),$$

$$\Omega1[\rho,z] \to \frac{\mathrm{Sin}[\theta]^2}{\Delta s[r,\theta]}\,\Sigma[r,\theta]\left(\frac{a\,(x+\bar{x})\,(4\,r\,M+\Delta s[r,\theta])}{\Delta s[r,\theta]}-a\,(y+\bar{y})\,\Delta[r]+\frac{i\,(z-\bar{z})\left(a^2\,\mathrm{Sin}[\theta]^2+\Delta[r]\right)\Sigma[r,\theta]}{\Delta s[r,\theta]\,\mathrm{Sin}[\theta]}\right)+\frac{a\,M\,fr[r,\theta]\,\mathrm{Sin}[\theta]^2\left(4a^2-6\,r\,M-4\,\Delta[r]+\Sigma[r,\theta]\right)}{\Delta s[r,\theta]^2}\Big\};$$

5 Outline

5 Conclusions and future prospects

- closed compact formula
- structure of discontinuities
- generalization of circular sources for $s = 2$
  - 6th derivatives of superpotential
  - it is working – compared results to Tagoshi, Sano
  - incoming/outgoing radiation gauge
  - more integration constants
  - construction of realistic rotating discs
- generalization of Kuzmin-Toomre
  - point particles on axis – cut
  - construction of realistic rotating discs

Thank you