To Kerr or not to Kerr?

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Despite the long history of efforts at spacelike singularity resolution in the classical GR solutions, the investigation of the non-singular, or so-called regular, black holes (RBH) and their regular rotating counterparts is extremely popular nowadays.

To construct a static regular solution, one relies on one of the following approaches:

- to solve Einstein's field equations associated with a special kind of spacetime symmetry and matter sources;
- to derive a solution as quantum corrections to the classical one;
- to write the metric *ad hoc*, motivating it by phenomenological "tractability", and try to analyze the effective matter content.

Nevertheless, figuring out the source sustaining the latter's appealing spacetimes is not trivial. In particular, one can regularize the Schwarzschild metric by replacing the radial coordinate, $u \to \sqrt{u^2 + b^2}$, resulting in a richer causal structure interpolating between a traversable wormhole and a black bounce geometry (Simpson & Visser, JCAP, 2019):

$$ds^2 = \left(1 - \frac{2m}{\sqrt{u^2+b^2}}\right)(dx^0)^2 - \frac{du^2}{1 - \frac{2m}{\sqrt{u^2+b^2}}} - (u^2+b^2)\,d\Omega_2^2.$$

This even one-parameter extension is sustained by a phantom scalar field and a magnetic field within nonlinear electrodynamics (Bronnikov & Walia, Phys.Rev.D, 2022).

The physical source for other black bounce spacetimes, not to mention numerous RBH models, is unknown. The only known thing about the sources in GR is the necessity to violate the energy dominance condition to avoid singularities (Hawking & Penrose, 1970).

Generalizing to realistic cases, rotating geometry, by imposing axial symmetry is more challenging: there are still only a few ways to introduce rotation into spacetime.

$\lozenge$ The first in origin is the Newman-Janis algorithm (NJA), which provides a set of steps to derive the axially symmetric solutions from the spherically symmetric ones:
· Schwarzschild $\rightarrow$ Kerr (Newman & Janis, J.Math.Phys., 1965);
· Reissner–Nordström $\rightarrow$ Kerr–Newman (Newman [et al.], J.Math.Phys., 1965).
The rigorous validity: the spacetime is an empty solution of Einstein's equations (EE) and belongs to the Kerr–Schild algebraic class (Schiffer [et al.], J.Math.Phys., 1973). However, the existence of the Kerr–Newman solution indicates that the first one is not necessary for NJA to be successful in general; this is established by the EE's fulfillment.

Notwithstanding, NJA is widely used to generate regular rotating solution disregarding either the source of a seed metric and/or its rotating counterpart.

$\lozenge$ One can generate a stationary solution by implication of an unknown function arising as a conformal metric multiplier, which explicit form is governed by the reducibility to the Boyer–Lindquist form and EE (Azreg-Aïnou, Eur.Phys.J.C, 2014 & Phys.Rev.D, 2014).

Consequently, this technique assumes that the physical source is known.

◊ A vacuum or electrovacuum axisymmetric solution can be obtained via the Ernst equation (Ernst, Phys.Rev. 167 & 168, 1968).

Its generalization to other cases is unknown.

Since the regularized GR solution requires a violation of energy dominance conditions, this may lead to interpolation between a regular black hole and a wormhole.

◊ The remaining solutions to the non-vacuum EE are known either perturbatively in a slow-rotation approximation (Kashargin & Sushkov, Grav.Cosmol. & Phys.Rev.D, 2008) or numerically (Kleihaus & Kunz, Phys.Rev.D, 2014; Chew,Kleihaus & Kunz, Phys.Rev.D, 2016)

In this talk:

We examined NJA on the so-called regular phantom black hole (Bronnikov & Fabris, Phys.Rev.Lett., 2006), whose geometry either provides a wormhole or a RBH with a Schwarzschild-like causal structure but with an asymptotically de Sitter expansion instead of a singularity.

This completely solvable example permits the matter's content via geometry.

Consider the following spherically symmetric metric:

$$ds^2 = A(u)\left(dx^0\right)^2 - \frac{du^2}{A(u)} - r^2(u)\left(\left(dx^2\right)^2 + \sin^2 x^2 \left(dx^3\right)^2\right); \quad u \in (-\infty; +\infty),$$

where $r(u)$ is regular, positive everywhere and has at least one minimum at some u_*:

$$r(u_*) > 0, \quad r'(u_*) = 0, \quad r''(u_*) > 0, \qquad \text{and} \qquad r(u) \sim |u| \quad \text{at} \quad u \to \pm\infty.$$

The exact solution can be derived for a minimally coupled scalar field with a wide set $L\big(\phi, (\phi_{,\mu})^2\big)$, which is able to violate the NEC: $T^\mu_\nu[\phi] k_\mu k^\nu \geq 0\,,\ \forall k^\mu : k_\mu k^\mu = 0$.

For $\phi = \phi(u)$, by noting $G^0_0 - G^2_2$ and choosing the simplest $r(u) = \sqrt{u^2 + b^2}$, we get

$$A(u) = 1 + c_1\left(u^2 + b^2\right) + c_2\left(\left(u^2 + b^2\right)\tan^{-1}\frac{u}{b} + ub\right).$$

Depending on c_1 and c_2 values, the obtained solution may be asymptotically flat or anti–de Sitter in the static region and asymptotically de Sitter in the nonstatic region.

By setting $c_1 = -\pi c_2/2$ and $c_2 b^3 \equiv u_0$ to ensure the regularity of $A(u)$ at $b \to 0$ and the Schwarzschild-like form, i.e., $A(u) \simeq 1 - 2u_0/3u$ at $u \to +\infty$, one obtains

$$A(u) = 1 - \frac{u_0}{b^3}\left(\left(u^2 + b^2\right)\cot^{-1}\frac{u}{b} - ub\right).$$

We have a traversable wormhole if $2b > \pi u_0$ (including as the exceptional case the Bronnikov–Ellis wormhole, $u_0 = 0$) or a RBH $\left(0 < 2b \leq \pi u_0\right)$ with a single horizon. Beyond the event horizon there is a bounce to anisotropic Kantowski–Sachs cosmology.

The Newman–Janis algorithm is based on introducing a vierbein of null vectors $\mathbf{e}_\alpha = \left(\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\right)$ and a series of complex conjugation transformations.

The steps are the following:

i) Switch to null (Eddington–Finkelstein) coordinates, $dx^0 \to d\tau = dx^0 - du/A(u)$, expressing the seed geometry as $ds^2 = \left(l_\mu n_\nu - m_\mu \bar{m}_\nu\right) dx^\mu dx^\nu$ via the null vectors

$$l^\mu = \delta^\mu_u, \qquad n^\mu = \delta^\mu_\tau - \frac{A(u)}{2}\delta^\mu_u, \qquad m^\mu = \frac{1}{\sqrt{2}r(u)}\left(\delta^\mu_2 + \frac{i}{\sin x^2}\,\delta^\mu_3\right);$$

ii) Complexify the seed metric functions, i.e., replace the redshift function $A(u)$ by a new one $A(u, \bar{u})$ and the area function $r(u)$ in complex null vectors m^μ and $\bar{m}^\mu$ as $r(u) = \sqrt{u^2 + b^2}$ and $\bar{r}(u) = \sqrt{\bar{u}^2 + b^2}$, requiring at $u = \bar{u}$ the recovery of initial $\mathbf{e}_\alpha$;

iii) Apply complex transformation coordinates, $x^\mu \to x'^\mu = x^\mu - ia\cos x^2\left(\delta^\mu_\tau - \delta^\mu_u\right)$, treating the primed coordinates as real with the tetrad transform, $\mathrm{e}^\mu_\alpha \to \mathrm{e}'^\mu_\alpha = \mathrm{e}^\nu_\alpha \frac{\partial x'^\mu}{\partial x^\nu}$, yielding the new $g'^{\mu\nu} = 2l'^{(\mu} n'^{\nu)} - 2m'^{(\mu}\bar{m}'^{\nu)}$ expression, and find an inverse metric;

iv) Revert the metric to the Boyer–Lindquist coordinates, which furnish only a single off-diagonal component, $g_{\tau x^3}$, throught an integrable coordinate transformation.

In our regular seed geometry' case, after steps i)—iii), we arrive at the inverse metric in the ingoing Eddington–Finkelstein coordinates being written via a line element

$$ds'^2 = A(u', x^{2'})\left(d\tau' - a\sin^2 x^{2'} dx^{3'}\right)^2 + 2\left(d\tau' - a\sin^2 x^{2'} dx^{3'}\right)\left(du' + a\sin^2 x^{2'} dx^{3'}\right)$$
$$-r\bar{r}(u', x^{2'})\left((dx^{2'})^2 + \sin^2 x^{2'}(dx^{3'})^2\right),$$

where

$$A(u', x^{2'}) = 1 + \frac{u_0 u'}{b^2} + \frac{a u_0 u'}{2b^3}\cos x^{2'} \ln\frac{u'^2 + (b - a\cos x^{2'})^2}{u'^2 + (b + a\cos x^{2'})^2} +$$
$$+ \frac{u_0}{2b^3}\left(u'^2 - a^2\cos^2 x^{2'} + b^2\right)\left(\tan^{-1}\frac{u'}{b + a\cos x^{2'}} + \tan^{-1}\frac{u'}{b - a\cos x^{2'}} - \pi\right)$$

and

$$r\bar{r}(u', x^{2'}) = \sqrt{(u'^2 - a^2\cos^2 x^{2'} + b^2)^2 + 4a^2 u'^2 \cos^2 x^{2'}}.$$

This obtained geometry does not contain Kerr's usual ring coordinate singularity at $u' = 0$ and $x^{2'} = \pi/2$, and it turns into the Kerr original one at the $b \to 0$ limit.

The curvature invariants for the obtained rotated solution are finite in the entire range of the u' coordinate: $R \sim (r\bar{r})^{-3}$ and $R_{\alpha\beta}R^{\alpha\beta} \sim R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim (r\bar{r})^{-6}$ are globally regular if $a \neq b$, and at the $b \to 0$ limit the standard Kerr' features are observed.

As the final step, it is not possible to find such integrable coordinate transformation, $d\tau' \to d\tau = d\tau' - \alpha(u')du'$ and $dx'^3 \to dx^3 = dx'^3 - \beta(u')du'$, that preserve $\alpha(u')$ and $\beta(u')$ independent of $x^{2'}$. Though we note that for a small regularizing parameter:

$$\alpha(u', x^{2'}) = \frac{g'^{\tau u}}{g'^{uu}} \simeq \frac{u'^2 + a^2}{u'^2 + a^2 - 2u_0 u'/3} + O(b^2),$$

$$\beta(u', x^{2'}) = \frac{g'^{ux^3}}{g'^{uu}} \simeq \frac{a}{u'^2 + a^2 - 2u_0 u'/3} + O(b^2),$$

these functions provide the Boyer–Lindquist transform; spacetime, being algebraically general, degenerates to an algebraically special and of Petrov type D up to $O(b^2)$.

In the slow-rotation approximation,

$$\alpha(u', x^{2'}) \simeq \frac{1}{A(u')} - \frac{a^2\left(1 - A(u')\right)}{A^2(u')r^2(u')} + O(a^4), \quad \beta(u', x^{2'}) \simeq \frac{a}{A(u')r^2(u')} + O(a^3),$$

our obtained geometry can also be reduced to the Boyer–Lindquist representation:

$$ds^2_{slow} \simeq \left(A(u') + O\left(a^2\right)\right)d\tau^2 + \left(2a\sin^2 x^{2'}(1 - A(u')) + O\left(a^3\right)\right)d\tau dx^3 - \\ - \left(A^{-1}(u') + O\left(a^2\right)\right)du'^2 - \left(r^2(u') + O\left(a^2\right)\right)\left((dx^{2'})^2 + \sin^2 x^{2'}(dx^3)^2\right),$$

coinciding at $u' \to +\infty$ with the slow rotation limit of the Kerr one up to $O\left(u'^{-3}\right)$ in these coordinates.

For a phantom scalar field, everything can be explicitly expressed via geometry. Hereafter, we apply a series of complex conjugation transformations in the NJ spirit.

As for the non-rotation case, the difference between $G^{\tau}_{\tau} (= G^0_0)$ and G^u_u, or G^{τ}_u itself, being matched with the stress-energy tensor of the scalar field, yields

$$-2\frac{r''}{r} = \epsilon \phi'^2 \quad \to \quad \phi_{\text{ph}}(u) = \pm\sqrt{2}\tan^{-1}\frac{u}{b} + \phi_0 = \sqrt{2}\cot^{-1}\frac{u}{b}.$$

The sum of G^{τ}_{τ} and G^u_u leads to an expression for potential via radial coordinate u:

$$V(u) = \frac{u_0\left((3u^2 + b^2)\cot^{-1}\frac{u}{b} - 3ub\right)}{b^3\left(u^2 + b^2\right)}.$$

One can reconstruct the exact expression for $V(u)$ via inverting $u = b\cot\frac{\phi_{\text{ph}}}{\sqrt{2}}$.

As for the rotating case, we complexified the scalar field, the potential, or Lagrangian density itself, i.e., replaced it and applied the complex transformation coordinates.

The non-trivial components of the resulted stress-energy tensor are asymtotically trivial at $u \to +\infty$, behave as $T^{\mu}_{\nu}[\phi_{\text{ph}}] \sim O(b^2)$ at the $b \to 0$ limit, and turn out coinciding with the exact non-rotation ones if $a = 0$.

However, the mixed Einstein tensor's components are all non-trivial and $G^\mu_\nu = T^\mu_\nu[\phi_{\mathrm{ph}}]$ are satisfied asymptotically, being noticeably violated only at distances on the order of the regularization parameter b, and $G^\mu_\nu \sim O(b^2)$ at $b \to 0$.
Since constructed geometry is Kerr' spacetime up to $O(b^2)$, this is guaranteed.

The null energy condition for a null vector k^μ, e.g., $k^\mu = \big(1/A(u,x^2), -1/2, 0, 0\big)$,

$$T^\mu_\nu k_\mu k^\nu = -\frac{1}{4}\big(\phi_{\mathrm{ph}}(u,x^2)\big)'^{\,2}_u = -\frac{b^2\big(u^2+b^2-a^2\cos^2 x^2\big)^2}{2\big(u^2+(b-a\cos x^2)^2\big)^2\big(u^2+(b+a\cos x^2)^2\big)^2}$$

is distinctly violated near an arbitrarily small region and slightly, $\sim O(b^2)$ at $u \to +\infty$.

Thereby, the EE's discrepancy and a violation of the energy dominance conditions are forced into this fairly small domain, for which parameter b is responsible. Among the known literature examples, only the "eye of the storm" (Simpson & Visser, JCAP, 2022) rotating RBH, being strictly a model, is similar in a sense of satisfying the classical energy dominance conditions at infinity for external observers.

Conclusion

- In spite of the RBH models' extreme popularity, only a few exact solutions are still known. To model realistic physical objects imposing axial symmetry, one will face the fact that there has not been elaborated a technique to generate a rotating solution from a static one.

  Most of these approaches are applicable either to vacuum cases, to linearized EE (which is the same as representing a metric in the Kerr–Schild form), or to exact solutions with a known physical source (which almost all RBH cannot boast of).

- We applied the mainstream NJ approach to regular static spacetime sustained by a phantom scalar field: at distances of the regularization parameter's order, we can predict or even conclude nothing due to the EE' discrepancies.

  Although coordinate complexification in the NJ spirit leads to a regular Kerr-like BH, to an external observer, this will be nothing more than Kerr's spacetime.

- Commonly, RBH' models *ad hoc* are motivated mainly by phenomenology, with possible observational verification. Many of them, or even pertubatively slowly rotating solutions in alternative theories, are almost indistinguishable from the GR solutions from an observational point of view (Psaltis [et al.], Phys.Rev.Lett., 2008; Pani & Cardoso, Phys.Rev.D, 2009; Shaikh [et al.], MNRAS, 2021).

  Moreover, the cost of simplicity and phenomenological "appealing", e.g., of the Simpson–Visser spacetime, are exotic sources.

Is it worth enforcing an exotic matter description for the static regular spacetimes, hereafter imposing axial symmetry approaches to rotating RBH models?

Or is the search for alternative GR singularity treatments more perspective?

Thank you for your attention!

Backup

$$T^{\mu}_{\nu}[\phi] = \epsilon \phi^{,\mu}\, \phi_{,\nu} - \frac{\delta^{\mu}_{\nu}}{2}\, \epsilon\, \phi^{,\alpha}\, \phi_{,\alpha} + \delta^{\mu}_{\nu} V(\phi), \qquad V(\phi_{\mathbf{ph}}) = \frac{u_0 \phi_{\mathbf{ph}}}{\sqrt{2} b^3}\left(3 - 2\sin^2 \frac{\phi_{\mathbf{ph}}}{\sqrt{2}}\right) - \frac{3u_0}{2b^3} \sin \sqrt{2}\phi_{\mathbf{ph}}$$

$$\phi_{\mathbf{ph}}(u, x^2) = \frac{\pi}{\sqrt{2}} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{b + a\cos x^2} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{b - a\cos x^2}$$

$$T^{\mu}_{\nu} = \begin{pmatrix} -L(u,x^2) & -\left(1 + \frac{a^2 \sin^2 x^2}{r\bar{r}(u,x^2)}\right) (\phi_{\mathrm{ph}})'^{\,2}_{u} & -\left(1 + \frac{a^2 \sin^2 x^2}{r\bar{r}(u,x^2)}\right) (\phi_{\mathrm{ph}})'_{u} (\phi_{\mathrm{ph}})'_{x^2} & 0 \\ 0 & \left(A(u,x^2) + \frac{a^2 \sin^2 x^2}{r\bar{r}(u,x^2)}\right) (\phi_{\mathrm{ph}})'^{\,2}_{u} - L(u,x^2) & \left(A(u,x^2) + \frac{a^2 \sin^2 x^2}{r\bar{r}(u,x^2)}\right) (\phi_{\mathrm{ph}})'_{u} (\phi_{\mathrm{ph}})'_{x^2} & 0 \\ 0 & \frac{(\phi_{\mathrm{ph}})'_{u} (\phi_{\mathrm{ph}})'_{x^2}}{r\bar{r}(u,x^2)} & \frac{(\phi_{\mathrm{ph}})'^{\,2}_{x^2}}{r\bar{r}(u,x^2)} - L(u,x^2) & 0 \\ 0 & -\frac{a(\phi_{\mathrm{ph}})'^{\,2}_{u}}{r\bar{r}(u,x^2)} & -\frac{a(\phi_{\mathrm{ph}})'_{u} (\phi_{\mathrm{ph}})'_{x^2}}{r\bar{r}(u,x^2)} & -L(u,x^2) \end{pmatrix},$$

where

$$\begin{aligned} L(u,x^2) = \frac{1}{b^2 (r\bar{r}(u,x^2))^4} \Big(& (u^2+b^2)^2 (3u_0 u^3 + 4u_0 b^2 u + b^4) + \big(u_0 u (9u^4 + 4b^2u^2 - b^4) - 2b^4 (3u^2 + b^2)\big) a^2 \cos^2 x^2 \\ & + (9u_0 u^3 - 6u_0 b^2 u + b^4) a^4 \cos^4 x^2 + 3u_0 u\, a^6 \cos^6 x^2 \Big) + \frac{u_0 u\, a\cos x^2}{2b (r\bar{r}(u,x^2))^2} \ln \frac{u^2 + (b - a\cos x^2)^2}{u^2 + (b + a\cos x^2)^2} \\ + \frac{u_0}{2b^3 (r\bar{r}(u,x^2))^2} & \Big((3u^2 + 2b^2)(u^2+b^2) + (6u^2 - 5b^2) a^2 \cos^2 x^2 + 3a^4 \cos^4 x^2 \Big) \left(\tan^{-1} \frac{u}{b + a\cos x^2} + \tan^{-1} \frac{u}{b - a\cos x^2} - \pi \right) \end{aligned}$$

In the slow-rotation limit $G^{\tau}_{3} \sim G^{u}_{\tau} \sim G^{u}_{3} \sim G^{2}_{\tau} \sim G^{2}_{3} \sim G^{3}_{\tau} \sim O(a^3)$, while other non-trivial components $\sim O(a^2)$.