POLARIZATIONS IN PHASE SPACE AND GRAVITATIONAL CHARGES

with Antoine Rignon-Bret and Simone Speziale

gloria odak
CPT Marseille

the role of boundary conditions

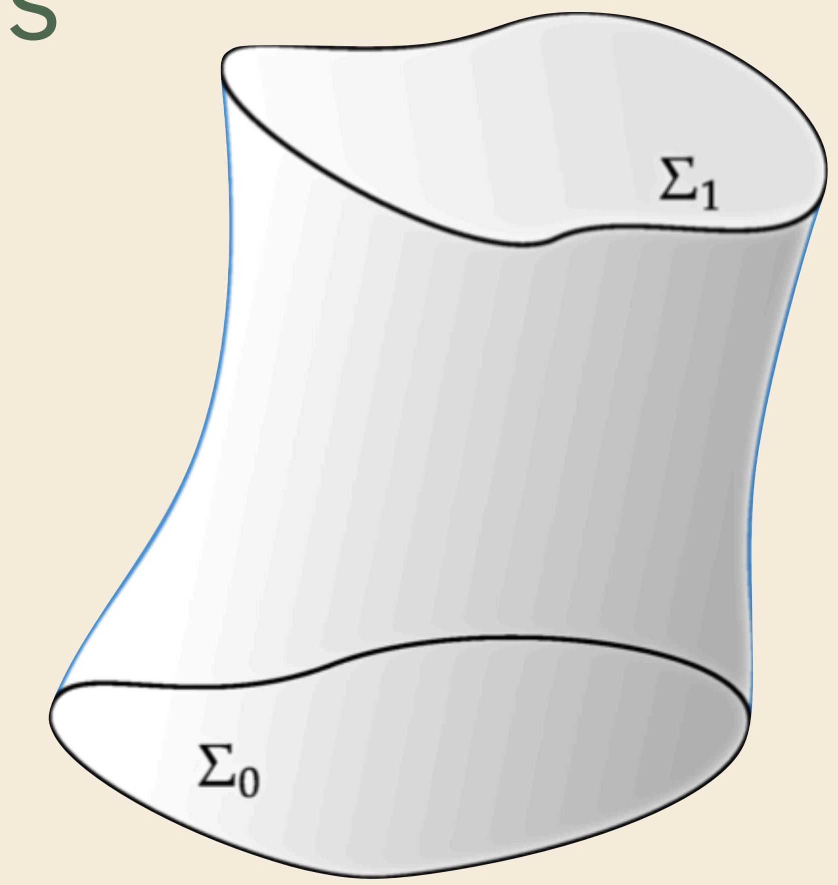
- Gravitational radiation makes the definition of energy ambiguous

the role of boundary conditions

- Gravitational radiation makes the definition of energy ambiguous
- This ambiguity is commonly fixed by imposing Dirichlet b.c.
    - Asymptotically flat
    - Quasilocal

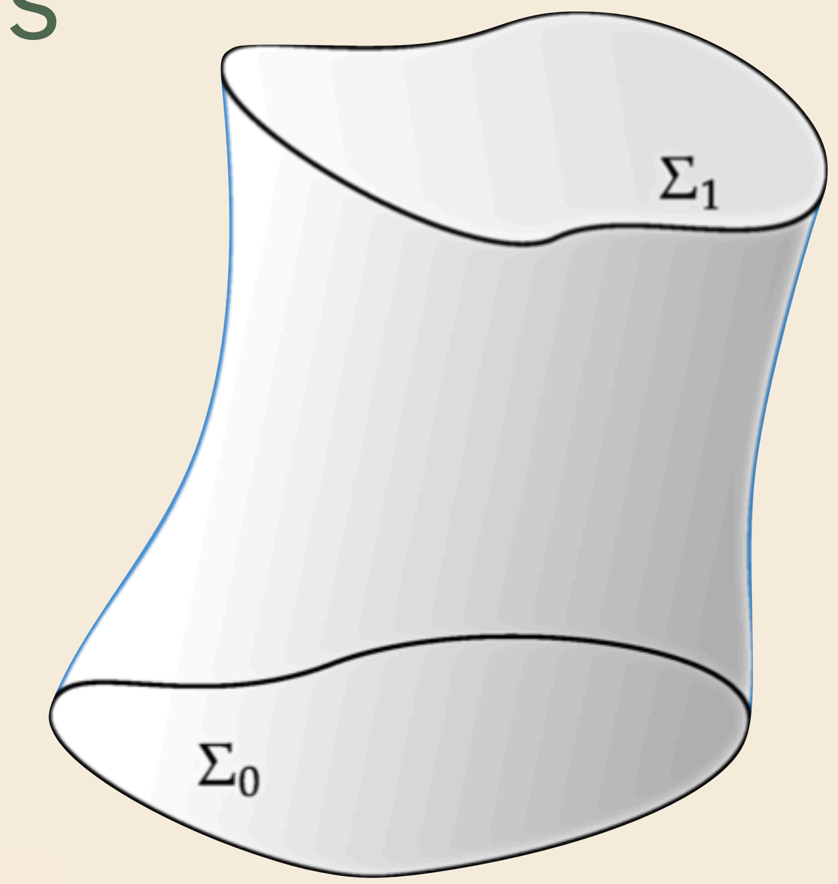
the role of boundary conditions

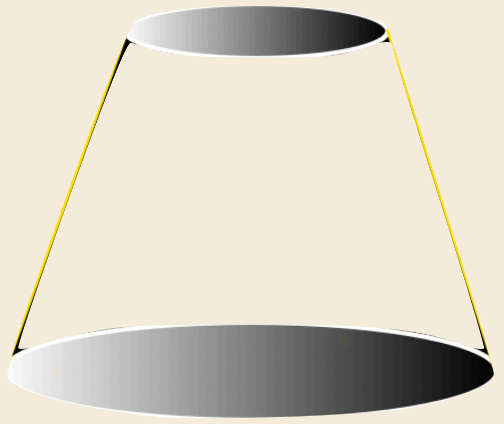
- Gravitational radiation makes the definition of energy ambiguous
- This ambiguity is commonly fixed by imposing Dirichlet b.c.
    - Asymptotically flat — ADM
    - Quasilocal — BY



the role of boundary conditions

- Gravitational radiation makes the definition of energy ambiguous
- This ambiguity is commonly fixed by imposing Dirichlet b.c.
    - Asymptotically flat — ADM, BMS
    - Quasilocal — BY

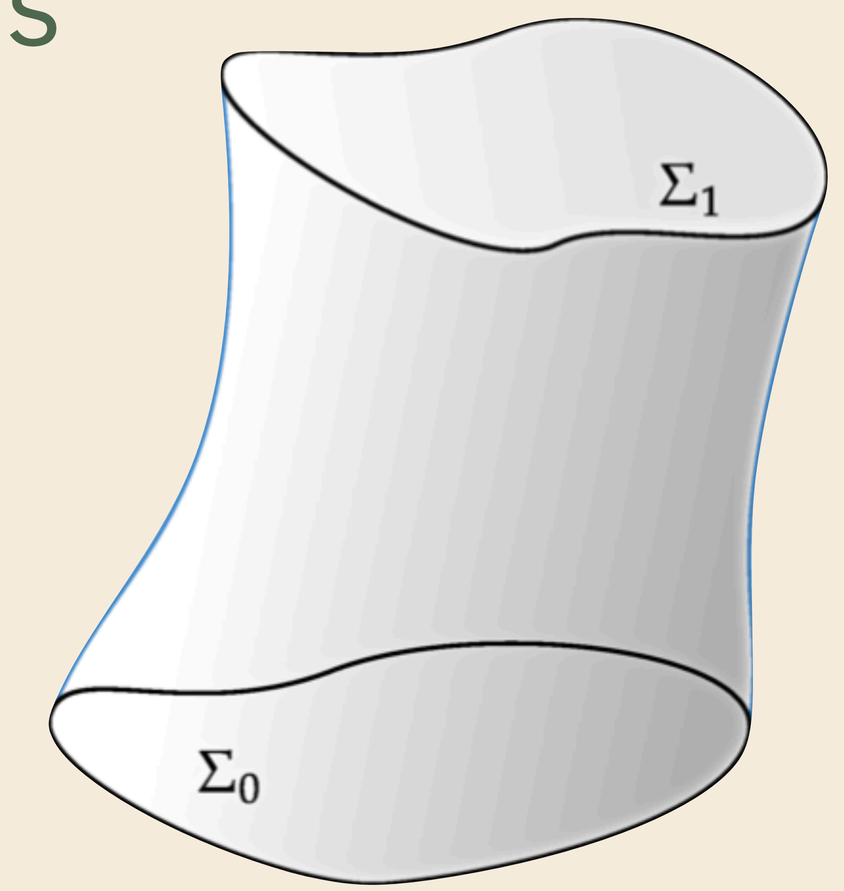


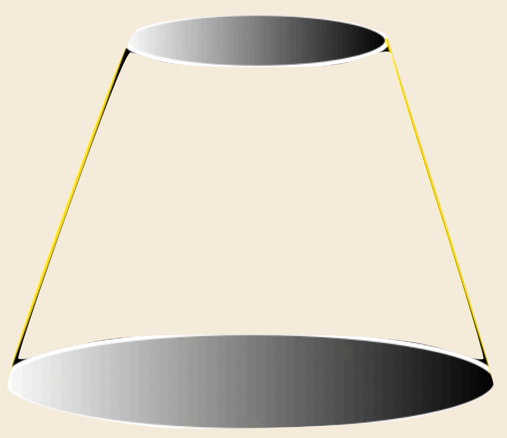


[Simone's talk on Friday]

the role of boundary conditions

- Gravitational radiation makes the definition of energy ambiguous
- This ambiguity is commonly fixed by imposing Dirichlet b.c.
  - Asymptotically flat — ADM, BMS
  - Quasilocal — BY
- Analogy with thermodynamics:
  - isothermal — internal energy
  - adiabatic — free energy

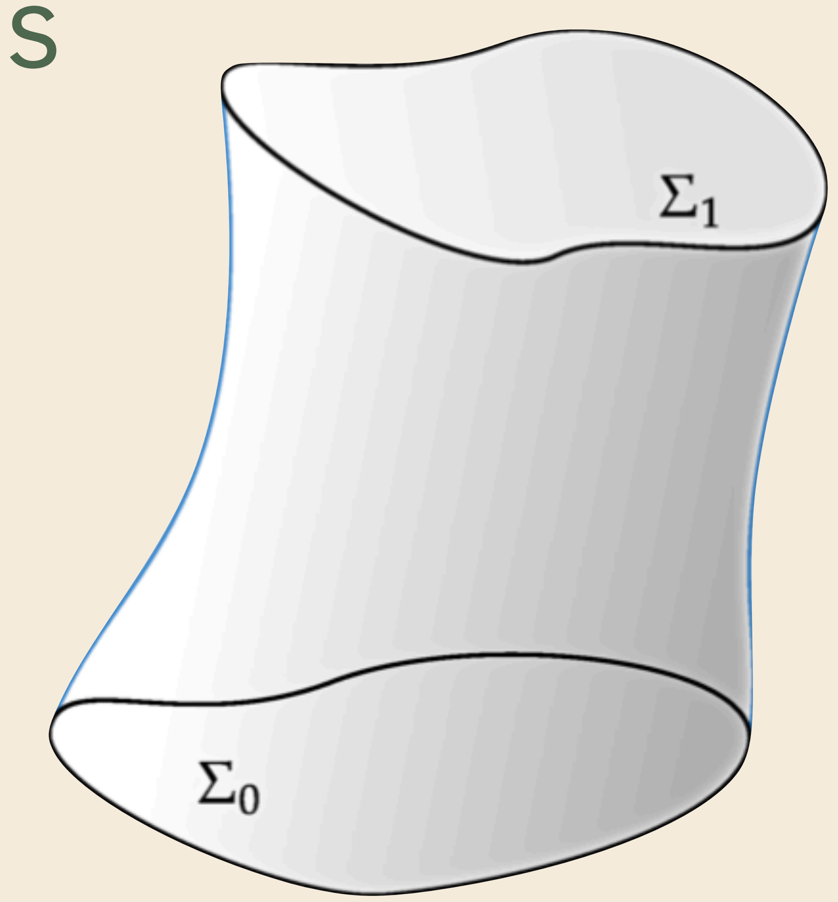


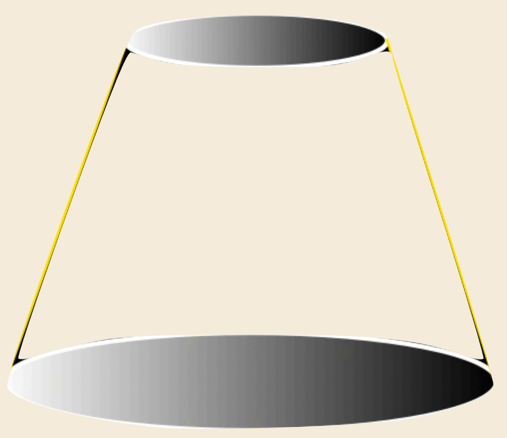


[Simone's talk on Friday]

the role of boundary conditions

- Gravitational radiation makes the definition of energy ambiguous
- This ambiguity is commonly fixed by imposing Dirichlet b.c.
    - Asymptotically flat — ADM, BMS
    - Quasilocal — BY
- Analogy with thermodynamics:
    - isothermal — internal energy
    - adiabatic — free energy
- How about gravity? What happens to energy if we use boundary conditions different than Dirichlet?





[Simone's talk on Friday]

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon, \quad d\theta :\approx \delta L$$

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon\theta - Y_\epsilon, \quad d\theta :\approx \delta L$$

↓

defined only up to corner terms

$\rightarrow j_\epsilon + da$ conserved as well

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon, \quad d\theta :\approx \delta L$$

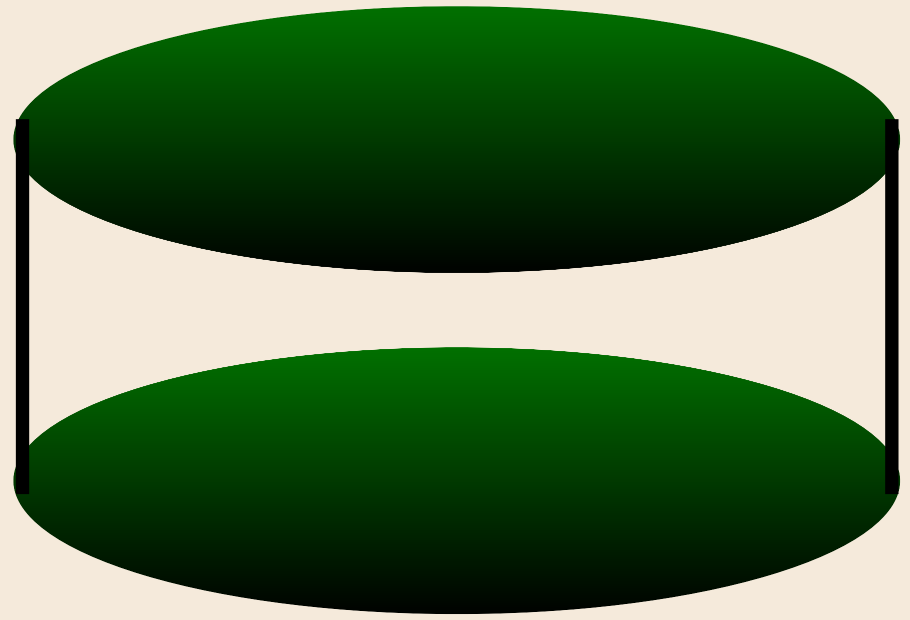
$$j_\epsilon = C_\epsilon + dq_\epsilon$$

= constraint (global charge) + boundary term (surface charge)

Noether's theorem

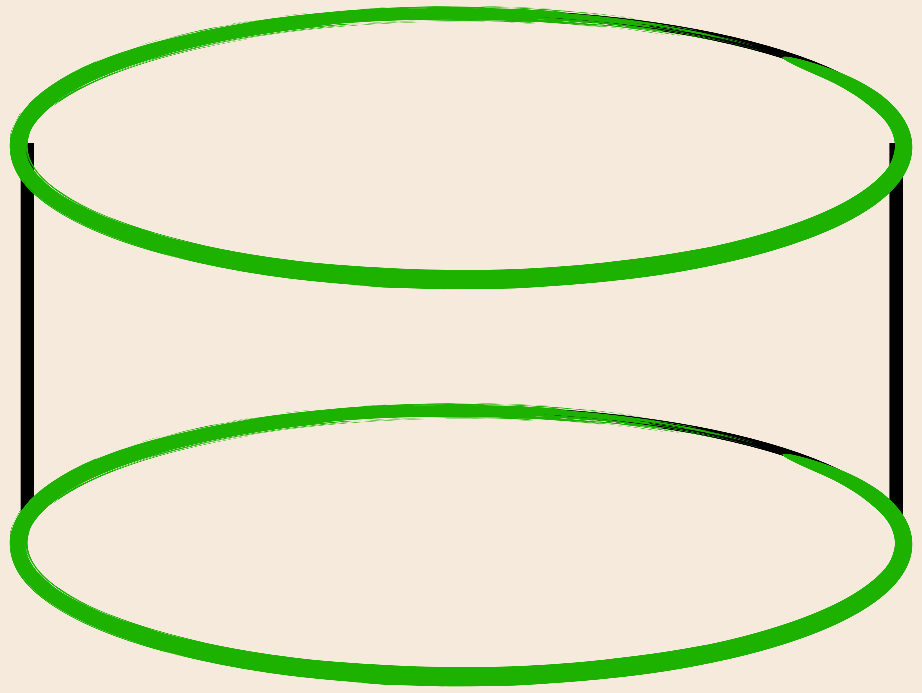
global example:

Poincaré invariance: conserved energy-momentum tensor



gauge version:

local diffeo invariance: conservation of surface charges



general relativity

Einstein-Hilbert: $L^{EH} = R\epsilon$

infinitesimal continuous symmetry: $\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \Rightarrow \delta_\xi L = d i_\xi L$

Noether current: $j_\xi = I_\xi \theta - Y_\xi = i_\xi E + d q_\xi$

$$j_\xi = G^\mu_\nu \xi^\nu - \nabla_\nu \nabla^{[\mu} \xi^{\nu]}$$

$$q_\xi = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma dx^\mu \wedge dx^\nu$$

general relativity

Einstein-Hilbert: $L^{EH} = R\epsilon$

infinitesimal continuous symmetry: $\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \Rightarrow \delta_\xi L = d i_\xi L$

Noether current: $j_\xi = I_\xi \theta - Y_\xi = i_\xi E + d q_\xi$

$$j_\xi = G^\mu_\nu \xi^\nu - \nabla_\nu \nabla^{[\mu} \xi^{\nu]}$$

$$q_\xi = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma dx^\mu \wedge dx^\nu$$

for Kerr:

$$\int_S q_{\partial_\varphi} = aM$$

$$\int_S q_{\partial_t} = \frac{M}{2}$$

scalar field with Neumann boundary conditions

- Consider the scalar field Lagrangian $\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$
- spacetime translation symmetry: $x^\nu \to x^\mu + \epsilon^\mu, \quad \delta_\epsilon \phi = -\epsilon^\mu \partial_\mu \phi,$
- $\delta \mathscr{L} = \Box \phi \delta \phi + \partial_\mu(\partial^\mu \delta \phi) \quad \to \quad \delta \phi \Big|_{\partial M} = 0, \quad \Box \phi = 0$ Dirichlet b.c.,

  $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha \phi \partial^\alpha \phi \quad \to E^D = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla \phi)^2$
- Neumann b.c.: $\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \partial_\mu(\phi \partial^\mu \phi) \quad \to \quad \delta \partial_n \phi \Big|_{\partial M} = 0, \quad \Box \phi = 0$

  $E^N = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla \phi)^2 - \partial_a(\phi \partial^a \phi) = E^D - \partial_a(\phi \partial^a \phi)$

boundary terms in the action of GR

- Start with the Einstein-Hilbert Lagrangian

  $$L^{EH} = R\epsilon$$

- Arbitrary variation gives $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d\left[\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma\right]$
- Need boundary Lagrangian $L = L^{EH} + d\ell$
- for Dirichlet b.c. it is Gibbons-Hawking-York $\ell^{GHY} = 2K\epsilon_\Sigma$
- what about different boundary conditions?

GR with Neumann boundary conditions

- $\delta L^{EH} \approx d\left[\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K\right)\epsilon_\Sigma\right]$
- the gravitational momentum $\tilde{\Pi}_{\mu\nu} := \sqrt{q}\left(K_{\mu\nu} - Kq_{\mu\nu}\right)$
- $\delta L^{EH} \approx d\left[\delta\tilde{\Pi}_{\mu\nu}q^{\mu\nu}\epsilon_\Sigma\right]$
- Neumann b.c. $\delta\tilde{\Pi}_{\mu\nu} = 0$
- no boundary Lagrangian required

York's mixed boundary conditions

- have 6 conditions to fix — can mix up Dirichlet and Neumann
- One particular choice of mixed b.c. that is geometrically motivated: Fixed conformal induced metric: $\hat{q}_{\mu\nu} := q^{-1/3} q_{\mu\nu}$ and the trace of extrinsic curvature $\delta \hat{q}_{\mu\nu} = 0 = \delta K$ [York '86]
- $\delta L^{EH} \approx d \left[-(P^{\mu\nu} \delta \hat{q}_{\mu\nu} + \frac{4}{3} \delta K) \epsilon_T - \frac{2}{3} \delta (K \delta q) \right]$
- boundary Lagrangian $\ell^Y = \frac{2}{3} K \epsilon_\Sigma$

All these cases can be parametrized by a real parameter b

$$\ell^b = bK\epsilon_\Sigma$$

$$L = L^{EH} + d\ell^b$$

boundary conditions	*quantity fixed on boundary*	*value of b*
Dirichlet	$q_{\mu\nu}$	2
York	$(\hat{q}_{\mu\nu}, K)$	2/3
Neumann	$\tilde{\Pi}^{\mu\nu}$	0

Noether charges for different boundary conditions

Dirichlet (b=2) $$q_\xi^{\mathrm{BY}} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S$$

[Brown, York '93]

York (b=2/3) $$q_\xi^{\mathrm{Y}} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S$$

Neumann (b=0) $$q_\xi^{\mathrm{N}} = -2\int_S n^\mu \xi^\nu \bar{K}_{\mu\nu}\epsilon_S$$

boundary conditions	*quantity held fixed*	*value of b*	*quasi-local energy*	*Kerr (renormalized)*
Dirichlet	$q_{\mu\nu}$	2	k	M
York	$(\hat{q}_{\mu\nu}, K)$	2/3	$k - 2\bar{K}/3$	$2M/3$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$k - \bar{K}$	$M/2$

[GO, Speziale '21]

Noether charges for different boundary conditions

Dirichlet (b=2) $$q_\xi^{\rm BY} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S$$

[Brown, York '93]

York (b=2/3) $$q_\xi^{\rm Y} = -2\int_S n^\mu \xi^\nu \left(\bar{K}_{\mu\nu} - \frac{1}{3}\bar{q}_{\mu\nu}\bar{K}\right)\epsilon_S$$

Neumann (b=0) $$q_\xi^{\rm N} = -2\int_S n^\mu \xi^\nu \bar{K}_{\mu\nu}\epsilon_S$$

boundary conditions	*quantity held fixed*	*value of b*	*quasi-local energy*	*Kerr (renormalized)*
Dirichlet	$q_{\mu\nu}$	2	k	M
York	$(\hat{q}_{\mu\nu}, K)$	2/3	$k - 2\bar{K}/3$	$2M/3$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$k - \bar{K}$	$M/2$

[GO, Speziale '21]

these charges are physically distinct because they correspond to canonical generators for different ways of making the system conservative

null boundaries

- the relation between different charges for different boundary conditions becomes clearer in the case of null boundaries

null boundaries

- known constraint-free data

null boundaries

- known constraint-free data
- don’t have to restrict to conservative b.c.

null boundaries

- known constraint-free data
- don’t have to restrict to conservative b.c.
- leaky b.c. not ambiguous

null boundaries

- known constraint-free data
- don't have to restrict to conservative b.c.
- leaky b.c. not ambiguous

Lehner, Myers, Poisson, Sorkin '16

Parattu, Chakraborty, Majhi, Padmanabhan '16

Donnay, Giribet, González, Pino '16

De Lorenzo, Perez '17

Hopfmuller, Freidel '18

Chandrasekaran, Flanagan, Prabhu '18

Oliveri, Speziale '19

Donnay, Marteau '19

Freidel, Oliveri, Pranzetti, Speziale '21

Chandrasekaran, Flanagan, Shehzad, Speranza '22

….

null boundaries

[GO, Rignon-Bret, Speziale WIP]

- canonical charges by Chandrasekaran, Flanagan, Prabhu '18 conserved on non-expanding horizons defined with Dirichlet polarization
- can use an analog of York's polarization on a null boundary — nice geometric propetries [Simone's talk on Friday]
- using this polarization leads to charges conserved both on NEH and on minkowski lightcones
- might have interesting implications for dynamical processes [Rignon-Bret '23]