On the classification of Generalized Quasitopological Gravities

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Although promising candidates exist (String Theory, Loop Quantum Gravity...), **Quantum Gravity remains** yet to be fully **understood**.

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What could we do?

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What could we do? We hope to study these phenomena by adding **suitable corrections** to GR...

But which ones?

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If $R_{abc}{}^{d}$ stands for Riemann curvature tensor and $R_{ac} = R_{abcd}{}^{b}$ for Ricci tensor, first-order corrections would be:

$$R^2 \,, \quad R_{ab}R^{ab} \,, \quad R_{abcd}R^{abcd} \,.$$

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In this presentation: **metric formalism** and **Levi-Civita connection**. However, there are other possibilities, like metric-affine theories **[*e.g.* Borunda, Janssen, Bastero-Gil '08; Olmo '11]**.

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- $\mathbf{f(R)}$ theories **[Buchdahl '70]**.

$$\mathcal{L}_{\mathrm{f(R)}} = R + f(R) \,,$$

for an arbitrary function f. If $f(R) = \alpha \ell^2 R^2$, we obtain **Starobinsky's model [Starobinsky '80]**.

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1. **Amenable** to **computations** (second-order or less eom under **certain circumstances**).
2. Non-trivial in **four dimensions**.
3. **Generic** enough so as to **capture** typical **features** introduced by higher-order terms.

Towards Generalized Quasitopological Gravities

Search **higher-order gravities** with second-order eom on single-function **static and spherically symmetric** (SSS) solutions:

$$\mathrm{d}s_f^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2\,.$$

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A theory is a **Generalized Quasitopological Gravity** *(GQG) if it admits single-function SSS solutions whose eom are second order.* **[Oliva, Ray '10; Myers, Robinson '10; Bueno, Cano '16; Hennigar, Kubizňák, Mann '17]**.

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GR and **Lovelock** gravities are **GQGs**. No non-trivial $f(R)$ is a GQG.

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$$\mathcal{L} = R + \alpha \ell^4 \mathcal{P}\,,$$
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- **Black hole thermodynamics** can be computed **analytically** **[*e.g.* Myers, Robinson '10; Bueno, Cano '16,'17; Hennigar, Kubizňák, Mann '17]**.
- ✔ **Any** purely gravitational higher-order **theory** can be **mapped** via perturbative **field redefinitions** to a **GQG** **[Bueno, Cano, Moreno, ÁM '19.]**

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This remains as outstanding open problem in literature. However, we have solved the problem in the class of **inequivalent GQGs**.

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In **our work [Moreno, ÁM '23]**, we **have found** the **explicit Lagrangians** of **all inequivalent GQGs** in $D \geq 4$. In this presentation, just show results in $D = 4$.

All inequivalent four-dimensional GQGs

Let W_{abcd} be Weyl tensor, $Z_{ab} = R_{ab} - \frac{1}{4} g_{ab} R$, $\pi_l = \mathrm{mod}(l, 2)$ and:

$$\mathcal{W}_l = \left(\frac{1}{3} W_{abcd} W^{abcd} \right)^{\frac{l-\pi_l}{2}} \left(\frac{1-\pi_l}{3} W_{abcd} W^{abcd} + \frac{2\pi_l}{3} W_{abcd} W^{cdef} W_{ef}{}^{ab} \right) .$$

Theorem (Moreno, Murcia '23)

The most general inequivalent GQG in $D = 4$ is

$$\mathcal{L} = R + \sum_{n=3}^{\infty} \alpha_n \ell^{2n-2} \mathcal{S}_{(n)} ,$$

where ℓ is a length scale, α_n arbitrary dimensionless constants and

$$\mathcal{S}_{(n)} = R^n - 6n(n-1) R^{n-2} Z_{ab} Z^{ab} + 18n(n-1)(n-2) R^{n-3} Z^{ab} Z^{cd} W_{abcd}$$
$$+ \sum_{l=0}^{n-2} \frac{(-3)^{l+2}(l+1)(3l+4)n!}{2(l+2)!(n-l-2)!} R^{n-l-4} \mathcal{W}_l \left(R^2 - \frac{48(n-l-2)(n-l-3)}{(l+1)(3l+4)} Z_{ab} Z^{ab} \right) .$$

Explicit expressions of generic GQGs

Lowest-order non-trivial GQGs for $D = 4$:

$$\mathcal{S}_{(3)} = R^3 + 18RW_{abcd}W^{abcd} - 36RZ_b^a Z_a^b - 126W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} + 108Z_b^a Z_d^c W_{ac}{}^{bd},$$

$$\begin{aligned}\mathcal{S}_{(4)} = {} & R^4 + 36R^2W_{abcd}W^{abcd} - 72R^2Z_b^a Z_a^b - 504RW_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} \\ & + 432RZ_b^a Z_d^c W_{ac}{}^{bd} + 135\left(W_{abcd}W^{abcd}\right)^2 - 216W_{abcd}W^{abcd}Z_f^e Z_e^f,\end{aligned}$$

$$\begin{aligned}\mathcal{S}_{(5)} = {} & R^5 + 60R^3W_{abcd}W^{abcd} - 120R^3Z_b^a Z_a^b - 1260R^2W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} \\ & + 1080R^2Z_b^a Z_d^c W_{ac}{}^{bd} + 675R\left(W_{abcd}W^{abcd}\right)^2 - 1080RW_{abcd}W^{abcd}Z_f^e Z_e^f \\ & - 1404W_{abcd}W^{abcd}W_{ef}{}^{gh}W_{gh}{}^{ij}W_{ij}{}^{ef} + 2160Z_b^a Z_a^b W_{cd}{}^{ef}W_{ef}{}^{gh}W_{gh}{}^{cd},\end{aligned}$$

Conclusions

- **GQGs** are higher-curvature extensions of Einstein's gravity:
  1. Admit **simple SSS solutions**.
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¡Muchas gracias!

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Any quantity **evaluated** on previous **single-function SSS metric** is said to be **on-shell**. Otherwise, **off-shell**.

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Throughout presentation: $\mathcal{Z}_{(n)}$ is order-n Quasitopological Gravity, $\mathcal{S}_{(n,j)}$ is an order-n proper GQG, with $j = 2, \ldots, n-1$ and $n \geq 1$.

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Throughout presentation: $\mathcal{Z}_{(n)}$ is order-n Quasitopological Gravity, $\mathcal{S}_{(n,j)}$ is an order-n proper GQG, with $j = 2, \ldots, n-1$ and $n \geq 1$.

The **on-shell expression** of all $n-1$ **inequivalent GQGs** in $D \geq 5$ is known **[Bueno, Cano, Hennigar, Lu, Moreno '22; Moreno, ÁM '23]**:

$$\mathcal{Z}_{(n)}|_f = \frac{1}{r^{D-2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{D-1}\left((2n-D)\tau_{(n,0)} - 2n\tau_{(n,1)}\right)\right] \, ,$$

$$\mathcal{S}_{(n,j)}|_f = \frac{1}{r^{D-2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{D-1}\left(\left(2 - \frac{D}{2n}(j+1)\right)\tau_{(n,0)} - (j+1)\tau_{(n,j)} + (j-1)\tau_{(n,j+1)}\right)\right] ,$$

where $\tau_{(n,k)} = (-f'/2)^k(1-f)^{n-k}r^{k-2n}$.

Classification of GQGs in $D \geq 5$

Since GQGs in $D \geq 5$ are fully characterized on-shell...$\rightarrow$ what if we develop a **dictionary** that allows us to **uplift on-shell expressions** to **off-shell** (covariant) expressions?

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In more general cases, how to **map on-shell** quantity to fully **covariant expression** with curvature tensors?

On-shell to off-shell dictionary

$$\mathrm{d}s_f^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2\,.$$

Consider orthogonal **projectors**:

$$T^a_b = \delta^a_t\delta^t_b + \delta^a_r\delta^r_b\,, \quad \sigma^a_b = g^a_b - T^a_b\,.$$

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It turns out **Weyl** tensor $W_{abc}{}^d$, **traceless Ricci** tensor $Z_{ab} = R_{ab} - \frac{R}{D}g_{ab}$:

$$W^{ab}{}_{cd}\big|_f = \Omega(r)\left[\frac{(D-2)(D-3)}{2}T^{[a}_{[c}T^{b]}_{d]} - (D-3)T^{[a}_{[c}\sigma^{b]}_{d]} + \sigma^{[a}_{[c}\sigma^{b]}_{d]}\right]\,,$$

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$$\Omega(r) = \frac{4 - 4f(r) + 4rf'(r) - 2r^2 f''(r)}{(D-1)(D-2)r^2} \, ,$$

$$\Theta(r) = \frac{2(D-3)(1-f(r)) + (D-4)rf'(r) + r^2 f''(r)}{Dr^2} \, .$$

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Proposition

Any contraction $(W^m Z^p R^q)_i$ of m Weyl tensors, p traceless Ricci tensors and q Ricci scalars satisfies:

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Goal: **Finding an appropriate basis** of **low-order terms** (i.e., low number of curvature tensors) so that **any contraction** of curvature tensors may be **obtained** by **multiplying** terms of this **basis**.

On-shell to off-shell dictionary

Define:

$$\mathsf{W}_2 \equiv \frac{4}{(D-2)^2(D-1)(D-3)} W_{abcd} W^{abcd} \,, \qquad \mathsf{Z}_2 \equiv \frac{2}{D(D-2)} Z^a_b Z^b_a \,,$$

$$\mathsf{W}_3 \equiv \frac{8}{(D-3)(D-2)(2(2-(D-3)^2)+(D-2)^2(D-3)^2)} W_{ab}{}^{cd} W_{cd}{}^{ef} W_{ef}{}^{ab} \,,$$

$$\mathsf{Y}_3 \equiv \frac{8}{D^2(D-2)(D-3)} Z^a_b Z^c_d W_{ac}{}^{bd} \,,$$

$$\mathsf{X}_3 \equiv -\frac{8}{(D-1)^2(D-2)(D-3)(D-4)} Z^a_b W_{acde} W^{bcde} \,,$$

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It turns out that:

$$\mathsf{W}_2\big|_f = \Omega^2\,, \quad \mathsf{Z}_2\big|_f = \Theta^2\,, \quad \mathsf{W}_3\big|_f = \Omega^3\,, \quad \mathsf{Y}_3\big|_f = \Theta^2\Omega\,,$$

$$\mathsf{X}_3\big|_f = \Omega^2\Theta\,, \quad \mathsf{Z}_3\big|_f = \Theta^3\,, \quad \mathsf{Y}_4\big|_f = \Theta^3\Omega\,, \quad \mathsf{X}_4\big|_f = \Omega^3\Theta\,.$$

On-shell to off-shell dictionary

Define:

$$\mathcal{I}_l^{(1)} = \mathsf{W}_2^{\frac{l-\pi_l}{2}} \left((1-\pi_l)\mathsf{W}_2 + \pi_l \mathsf{W}_3 \right) \,, \quad \mathcal{I}_l^{(2)} = \mathsf{Z}_2^{\frac{l-\pi_l}{2}} \left((1-\pi_l)\mathsf{Z}_2 + \pi_l \mathsf{Z}_3 \right) \,,$$
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with $\pi_l = \mathrm{mod}(l, 2)$.

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On-shell		Off-shell
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$\mathrm{P}(r)$	$\longrightarrow$	$R\,.$

The terms $\mathrm{P}^p\Omega$, $\mathrm{P}^p\Theta$ and $\mathrm{P}^p\Theta\Omega$ alone cannot be translated into off-shell quantities.

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$$\mathcal{Z}_{(2)}|_f = \frac{1}{r^{D-2}} \frac{\mathrm{d}}{\mathrm{d}r} \left[\left(\frac{1-f}{r^2} \right)^n r^{D-1} \left(\frac{(4-D)(1-f)}{r^2} + \frac{2f'}{r} \right) \right] .$$

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3. Third step (after rescaling $\mathcal{Z}_{(2)}$ so that R^2 is normalized to one):

$$\mathcal{Z}_{(2)} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} .$$

All inequivalent Quasitopological Gravities in $D \geq 5$

Theorem

The **unique inequivalent Quasitopological Gravity** *at each curvature order* $n \geq 3$ *for* $D \geq 5$ *can be chosen to be*

$$\begin{aligned}\mathcal{Z}_{(n)} = R^n &+ \sum_{l=0}^{n-2} R^{n-l-2}\left(\gamma_{n,-2,l}\mathcal{I}_l^{(1)} + \gamma_{n,l,-2}\mathcal{I}_l^{(2)}\right)\\ &+ \sum_{l=0}^{n-3} R^{n-l-3}\left(\gamma_{n,-1,l}\mathcal{I}_l^{(3)} + \gamma_{n,l,-1}\mathcal{I}_l^{(4)}\right)\\ &+ \sum_{l=0}^{n-4}\sum_{p=0}^{n-l-4} \gamma_{n,l,p} R^{n-l-p-4}\mathcal{I}_p^{(1)}\mathcal{I}_l^{(2)}\,, \quad n \geq 3\,,\end{aligned}$$

with constants $\gamma_{n,l,p}$ *only non-zero for* $l, p \geq -2$ *and* $l + p + 4 \leq n$, *in which case*

$$\gamma_{n,l,p} = \frac{n!(D(D(l-2)+4)(l+1) + 4(D-1)(Dl+1)(p+2) + 4(D-1)^2(p+2)^2)}{2^{2-l+p}(D^2-D)^{-p-l-3}(D-2)^{l+2}(l+2)!(p+2)!(n-l-p-4)!}\,.$$

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- For $n = 1$ and $n = 2$, we only have GR and Gauss-Bonnet.

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- For $n = 1$ and $n = 2$, we only have GR and Gauss-Bonnet.
- It **drastically simplifies** previous expressions for Quasitopological Gravities.

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The $n-2$ **inequivalent proper GQGs** *at each curvature order* $n \geq 3$ *with* $D \geq 5$ *can be taken to be*

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$$\begin{aligned}\mathcal{S}_{(n,j)} = R^n &+ \sum_{l=0}^{n-2} R^{n-l-2}\left(\sigma_{n,j,-2,l}\mathcal{I}_l^{(1)} + \sigma_{n,j,l,-2}\mathcal{I}_l^{(2)}\right)\\ &+ \sum_{l=0}^{n-3} R^{n-l-3}\left(\sigma_{n,j,-1,l}\mathcal{I}_l^{(3)} + \sigma_{n,j,l,-1}\mathcal{I}_l^{(4)}\right)\\ &+ \sum_{l=0}^{n-4}\sum_{p=0}^{n-l-4} \sigma_{n,j,l,p} R^{n-l-p-4}\mathcal{I}_p^{(1)}\mathcal{I}_l^{(2)}\,, \quad n \geq 3\,,\end{aligned}$$

with $j = 2, \ldots, n-1$ *and* $\sigma_{n,j,l,p}$ *given prescribed constants.*

- For $n = 1$ and $n = 2$, we only have GR and Gauss-Bonnet.
- First classification in the literature of **all inequivalent GQGs** in $D \geq 5$.

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Conclusion: **All contractions** with **odd** numbers of Z^a_b **vanish** identically on single-function SSS ansatz in $D = 4$.

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Dictionary:

On-shell		**Off-shell**
$\Omega(r)^{l+2}$	$\longrightarrow$	$\mathcal{I}_l^{(1)}$
$\Theta(r)^{2l+2}$	$\longrightarrow$	Z_2^{1+l}
$\Omega(r)\Theta(r)^{2l+2}$	$\longrightarrow$	$\mathsf{Z}_2^{l}\mathsf{Y}_3 \,,$
$\mathrm{P}(r)$	$\longrightarrow$	$R \,,$

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Observe that $P^q\Omega$ and $P^q\Omega^s\Theta^{2l+1}$ alone **cannot** be translated into off-shell quantities.

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$$\mathcal{F}_n = \alpha_n \mathcal{Z}_{(n)}|_f + \sum_{j=2}^{n-1} \beta_{n,j} \mathcal{S}_{(n,j)}|_f \,, \quad n \geq 3 \,,$$

for arbitrary coefficients $\alpha_n, \beta_{n,j}$.

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Result: It **happens if and only** if $\alpha_n = \beta_{n,j} = 0 \,, \quad j = 2, \ldots, n-2$. In fact:

$$\mathcal{F}_n = \mathcal{S}_{(n,n-1)}|_f = \frac{3}{12^n} (\mathrm{P} - 3\Omega)^{n-2} \left(-2\mathrm{P}^2 - 6(n-2)\mathrm{P}\Omega \right.$$
$$\left. +3(n-1)(16n\Theta^2 + 3(2-3n)\Omega^2) \right) \,,$$

All GQGs in $D = 4$

Theorem

There exists a **unique inequivalent GQG** *at each curvature order* $n \geq 3$ *in* $D = 4$. *It can be taken to be*

$$\mathcal{S}_{(n)}^{(4)} = R^n + \gamma_1 R^{n-2} \mathsf{Z}_2 + \gamma_2 R^{n-3} \mathsf{Y}_3 + \sum_{l=0}^{n-2} \lambda_l^{(1)} R^{n-l-4} \mathcal{I}_l^{(1)} \left(R^2 + \lambda_l^{(2)} \mathsf{Z}_2 \right) ,$$

where

$$\gamma_1 = -24n(n-1) \,, \quad \gamma_2 = -3(n-2)\gamma_1 \,, \quad \lambda_l^{(1)} = \frac{(-3)^{l+2}(l+1)(3l+4)n!}{2(l+2)!(n-l-2)!} \,,$$
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- For $n = 1$ we just have GR, for $n = 2$ there is no non-trivial GQG.
- **First proof** of fact that there is **one and only one inequivalent GQG** at each order in $D = 4$.

Examples of GQGs

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- **Cubic Quasitopological Gravity** [Oliva, Ray '10; Myers, Robinson '10]:

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\begin{aligned}
\mathcal{Z}_{(3)} =& R_a{}^b{}_c{}^d R_b{}^e{}_d{}^f R_e{}^a{}_f{}^c + \frac{1}{(2D-3)(D-4)}\left[\frac{3(3D-8)}{8} R_{abcd}R^{abcd}R \right. \\
& -\frac{3(3D-4)}{2} R_a{}^c R_c{}^a R - 3(D-2) R_{acbd}R^{acb}{}_e R^{de} + 3D R_{acbd}R^{ab}R^{cd} \\
& \left. +6(D-2) R_a{}^c R_c{}^b R_b{}^a + \frac{3D}{8}R^3\right] .
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It defines a Quasitopological Gravity for $D \geq 5$ (algebraic eom for f).

No Quasitopological Gravity exists for $D = 4$ (apart from GR).

Examples of GQGs

- **Cubic proper GQG** [Bueno, Cano '16, Hennigar, Kubizňák, Mann '17]:

$$\mathcal{S}_{(3)} = 14 R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + 2 R_{ab}^{cd} R_{ce}^{ab} R_{d}^{e} - \frac{(38 - 29D + 4D^2)}{4(D-2)(2D-1)} R_{abcd} R^{abcd} R$$
$$- \frac{2(4D^2 + 9D - 30)}{(D-2)(2D-1)} R_{abcd} R^{ac} R^{bd} - \frac{4(2D^2 - 35D + 66)}{3(D-2)(2D-1)} R_a^b R_b^c R_c^a$$
$$+ \frac{(4D^2 - 21D + 34)}{(D-2)(2D-1)} R_{ab} R^{ab} R - \frac{(4D^2 - 13D + 30)}{12(D-2)(2D-1)} R^3 \,.$$

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It is a **non-trivial GQG** for every $D \geq 4$ (second-order eom for f). In $D = 4$, by combining with cubic Lovelock density:

$$\mathcal{P} = 12 R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + R_{ab}^{\ \ cd} R_{cd}^{\ \ ef} R_{ef}^{\ \ ab} - 12 R_{abcd} R^{ac} R^{bd} + 8 R_a^b R_b^c R_c^a \,,$$

This is **Einsteinian Cubic Gravity** (ECG), first proper GQG identified.