On the Canonical Equivalence of the Jordan and Einstein Frames

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Jordan-Einstein Frames

- Old paper: Dicke (Phys. Rev. (1962) **125,** 6 2163-2167)

Suppose the proton mass is m_p in mass units m_u and, in "natural units", we scale the unit of measurement by a factor λ^{-1} (length)$^{-1}$
$\tilde{m}_u = \lambda^{-1} m_u$. In the new unit the proton mass $\tilde{m}_p = \lambda^{-1} m_p$.

- Confronting the measurement of the proton mass in the two mass units (Faraoni and Nadeau 2007)

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\lambda^{-1} m_p}{\lambda^{-1} m_u} = \frac{m_p}{m_u}$$

Jordan-Einstein Frames

- Since $d\tilde{s} = \lambda ds$ and $ds = \left(g_{ij} dx^i dx^j\right)^{\frac{1}{2}}$, then the covariant metric functions scales as

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

- Invariance under rescaling of units of measurement implies Weyl (conformal) invariance of the metric tensor

- The starting frame is called "Jordan" frame and the conformal transformed the "Einstein Frame. One observable can be computed in both frames. Its measure, obviously different in the two frames, is related by conformal rescaling according to the observable's dimensions.(e.g. $\tilde{m}_p = \lambda^{-1} m_p$).

Scalar-Tensor Theory

- In general, one starts from a scalar-tensor theory, with GHY-like boundary term, in the Jordan Frame

$$S = \int_M d^n x \sqrt{-g} \left(f(\phi) R - \frac{1}{2} \lambda(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + 2 \int_{\partial M} d^{n-1} \sqrt{h} f(\phi) K$$

- and passes to the Einstein Frame with the transformation

$$\tilde{g}_{\mu\nu} = \left(16\pi G f(\phi)\right)^{\frac{2}{n-2}} g_{\mu\nu} \ ,$$

- therefore, the action becomes

$$S = \int_M d^n x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G} \tilde{R} - A(\phi) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \frac{1}{8\pi G} \int_{\partial M} d^{n-1} \sqrt{\tilde{h}} \tilde{K}$$

$$A(\phi) = \frac{1}{16\pi G} \left(\frac{\lambda(\phi)}{2 f(\phi)} + \frac{n-1}{n-2} \frac{(f'(\phi))^2}{f^2(\phi)} \right), V(\phi) = \frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}}$$

- It is assumed that if $(g_{\mu\nu}(x), \phi(x))$ is solution of the E.O.M also $(\tilde{g}_{\mu\nu}(x, \phi), \phi(x))$ is solution (True?). This reasoning seems to address that the transformation from the Jordan to the Einstein frame look like a canonical transformation in the Hamiltonian theory.

Brans-Dicke Theory

- Brans-Dicke, with GHY boundary term, is a particular case of Scalar Tensor theory ($f(\phi) = \phi$)

$$S = \int_M d^4x\sqrt{-g}\left(\phi\,{}^4R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K \quad .$$

- How to perform canonical analysis of this theory?

Garay and Gracìa-Bellido NPB 400 (1993): the transformations are Hamiltonian canonical.

$$\bar{q}_{ik} = \Phi q_{ik}, \quad \phi = -\frac{1}{2\beta}\log \Phi$$

$$\bar{N}^2 = \Phi N^2, \quad \bar{N}_i = \Phi N_i$$

Deruelle, Sendouda, Youssef PRD 80, (2009). They still claim that the transformations are Hamiltonian canonical

$$\tilde{h}_{ab} = \phi h_{ab}, \quad \tilde{N}^a = N^a, \quad \tilde{N} = \sqrt{\phi}N, \quad \tilde{\phi} = \sqrt{\frac{3}{2}}\ln\phi$$

$$\tilde{p}^{ab} = \frac{1}{\phi}p^{ab}, \quad \tilde{\pi} = \sqrt{\frac{2}{3}}(\phi\pi - p)$$

$$\{\tilde{h}_{ab}, \tilde{p}^{cd}\}_J = \{h_{ab}, p^{cd}\}_J, \ \{\tilde{\phi}, \tilde{\pi}\}_J = \{\phi, \pi\}_J, \quad \{\tilde{p}^{ab}, \tilde{\pi}\}_J = 0, \quad \{\tilde{h}_{ab}, \tilde{\phi}\}_J = 0, \quad \{\tilde{h}_{ab}, \tilde{\pi}\}_J = 0$$

$$\{\tilde{p}^{ab}, \tilde{\phi}\}_J = 0$$

Brans-Dicke Theory

- The Hamiltonian Weyl (conformal) transformations from the Jordan to the Einstein frames are

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}}\,;\tilde{N}_i = N_i(16\pi G\phi)\,;\tilde{h}_{ij} = (16\pi G\phi)\,h_{ij}\,;\tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}}\,;$$

$$\tilde{\pi}^i = \frac{\pi^i}{(16\pi G\phi)},;\tilde{\pi}^{ij} = \frac{\pi^{ij}}{16\pi G\phi}\,;\phi = \phi\,;\tilde{\pi}_\phi = \frac{1}{\phi}(\phi\pi_\phi - \pi_h)$$

- They are not Hamiltonian canonical

$$\{\tilde{N}, \tilde{\pi}_\phi\} = \frac{8\pi GN}{\sqrt{16\pi G\phi}} \neq 0, \text{and } \{\tilde{N}_i, \tilde{\pi}_\phi\} = 16\pi GN_i \neq 0$$

- The Dirac's constraint analysis of the Hamiltonian theory has to be done, independently, in the Jordan and Einstein frames. We have studied the Hamiltonian constrained theory in Jordan and Einstein frames for both cases $\omega \neq -\frac{3}{2}$ and , $\omega = -\frac{3}{2}$. In the case $\omega = -\frac{3}{2}$ the theory has an extra Weyl(conformal) symmetry with an associated primary first class constraint C_ϕ

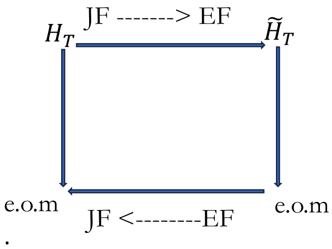
Hamiltonian Analysis of BD for $\omega \neq -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi \approx 0; \pi^i \approx 0; \mathcal{H} \approx 0; \mathcal{H}_i \approx 0;$	*constraints* $\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{\mathcal{H}} \approx 0; \widetilde{\mathcal{H}}_i \approx 0;$
constraint algebra $\{\pi, \pi_i\} = 0; \{\pi, \mathcal{H}\} = 0; \{\pi, \mathcal{H}_i\} = 0; \{\pi_i, \mathcal{H}\} = 0;$ $\{\pi_i, \mathcal{H}_j\} = 0;\ \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial'_i\delta(x,x');$ $\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x,x') - \mathcal{H}_j(x)\partial_i{}'\delta(x,x');$ $\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x,x') - \mathcal{H}^i(x')\partial'_i\delta(x,x');$	*constraint algebra* $\{\widetilde{\pi}, \widetilde{\pi}_i\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i\} = 0; \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j\} = 0;\ \left\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}_i(x')\right\} = -\widetilde{\mathcal{H}}(x')\partial'_i\delta(x,x');$ $\{\widetilde{\mathcal{H}}_i(x), \widetilde{\mathcal{H}}_j(x')\} = \widetilde{\mathcal{H}}_i(x')\partial_j\delta(x,x') - \widetilde{\mathcal{H}}_i(x)\partial_i{}'\delta(x,x');$ $\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}(x')\} = \widetilde{\mathcal{H}}^i(x)\partial_i\delta(x,x') - \widetilde{\mathcal{H}}^i(x')\partial'_i\delta(x,x');$

Hamiltonian Analysis of BD for $\omega = -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi_N \approx 0; \pi^i \approx 0; C_\phi \approx 0; \mathcal{H}^{(-3/2)} \approx 0; \mathcal{H}_i^{(-3/2)} \approx 0;$	*constraints* $\widetilde{\pi}_N \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{C}_\phi = -\widetilde{\phi}\widetilde{\pi}_\phi \approx 0; \widetilde{\mathcal{H}}^{(-3/2)} \approx 0; \widetilde{\mathcal{H}}_i^{(-3/2)} \approx 0;$
constraint algebra $\{\pi_N, \pi_i\} = \{\pi_N, \mathcal{H}^{(-3/2)}\} = \{\pi_N, \mathcal{H}_i^{(-3/2)}\} = 0;$ $\{\pi_i, \mathcal{H}^{(-3/2)}\} = \{\pi_i, \mathcal{H}_j^{(-3/2)}\} = 0;$ $\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial'_i\delta(x,x')C_\phi(x');$ $\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x,x');$ $\left\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\mathcal{H}^{(-3/2)}(x')\partial'_i\delta(x,x');$ $\left\{\mathcal{H}_i^{(-3/2)}(x), \mathcal{H}_j^{(-3/2)}(x')\right\} = \mathcal{H}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\mathcal{H}_j^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} =$ $\mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x,x') +$ $\left[D^i(\log\phi(x))\right] C_\phi(x)\partial_i\delta(x,x')$ $-\left[D^i(\log\phi(x'))\right] C_\phi(x')\partial'_i\delta(x,x');$	*constraint algebra* $\{\widetilde{\pi}_N, \widetilde{\pi}_i\} = \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}^{(-3/2)}\} = 0; \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}_i^{(-3/2)}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}^{(-3/2)}\} = \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j^{(-3/2)}\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = -\widetilde{\mathcal{H}}^{(-3/2)}(x')\partial'_i\delta(x,x');$ $\{\widetilde{\mathcal{H}}_i^{(-3/2)}(x), \widetilde{\mathcal{H}}_j^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} =$ $\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial'_i\delta(x,x');$

FLAT FLRW Brans-Dicke theory

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^3$$

$$\mathcal{L}_{FLRW} = -\frac{6a\dot{a}^2}{N}\phi - \frac{6a^2\dot{a}}{N}\dot{\phi} + \frac{\omega a^3}{N\phi}(\dot{\phi})^2 - Na^3U(\phi)$$



$$\dot{N} \approx \lambda_N \, , \tag{1}$$

$$\dot{\pi}_N = -H \approx 0 \, , \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \, , \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \, , \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \, , \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} \tag{6}$$

$$\dot{N} \approx \frac{\widetilde{\lambda}_N}{(16\pi G\phi)^{\frac{1}{2}}} - \frac{N^2}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \, , \tag{1}$$

$$\dot{\pi}_N \approx -H + \frac{N\pi_N}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \, , \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \, , \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \, , \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \, , \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} + \frac{H}{2\phi} \, . \tag{6}$$

CANONICAL EQUIVALENCE OF JF AND EJ VIA GAUGE FIXING

- We have performed the following gauge fixing in the Jordan Frame and in the Einstein Frame

$$\text{Jordan Frame } N \approx c \, , N_i \approx c_i \;\; \mapsto \;\; \text{Einstein Frame } \widetilde{N} - c(16\pi G\phi)^{\frac{1}{2}} \approx 0 \, , \widetilde{N}_i - c_i(16\pi G\phi) \approx 0$$

- The secondary first class constraints $\pi \approx 0$ and $\pi_i \approx 0$ become second class constraints

- It is possible to define Dirac's brackets and solve the second class constraints

$$\{\, , \,\}_{DB} \equiv \{\, , \,\} - \{\, , \varphi_\alpha\}C^{-1}_{\alpha\beta}\{\varphi_\beta, \,\} \;\; C_{\alpha\beta} \equiv \{\varphi_\alpha, \varphi_\beta\} \text{ being } \varphi_\alpha, \varphi_\beta \text{ second class contraints}$$

- The transformations from the Jordan to the Einstein frames result to be Hamiltonian canonical transformations. Remember: now the phase space is a reduced one, where we have gauge-fixed the lapse function N and the shift functions N_i .

- Does it mean that the two frames are physically equivalent?

CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- Harmonic Oscillator (Goldstein)

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

- Canonical transformations (not symmetry of the system…)

$$q = \sqrt{\frac{2P}{m\omega}} sinQ \,, p = \sqrt{2m\omega P} cosQ$$

- Therefore the Hamiltonian becomes

$$H = \omega P$$

- and then,

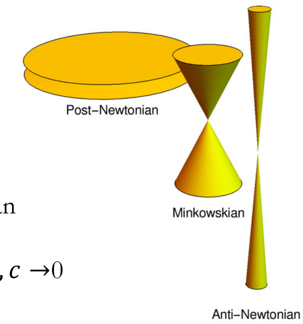
$$P = \frac{E}{\omega}, \ \dot{Q} = \frac{\partial H}{\partial P} = \omega, \ Q = \omega t + \alpha, \ q(t) = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha)$$

- Notice that the harmonic oscillator is mapped into a free particle

ANTI-GRAVITY TRANSFORMATIONS (Canonical Transformations)

- There exist Hamiltonian Canonical Transformations on the extended phase space: The Anti-Gravity transformations

$$\widetilde{N}^* = N \ ; \widetilde{\pi}_{N^*} = \pi_N \ ; \widetilde{N}_i^* = N_i \ ; \widetilde{\pi}^{*i} = \pi^i \ ; \widetilde{h}_{ij}^* = (16\pi G\phi) h_{ij} \ ;$$
$$\widetilde{\pi}^{*ij} = \frac{\pi^{ij}}{(16\pi G\phi)^{\frac{1}{2}}} \ ; \widetilde{\phi}^* = \phi \ ; \widetilde{\pi}_\phi^* = \frac{1}{\phi} (\phi\pi_\phi - \pi_h) \ ;$$



Carrolian Gravity, G→ ∞ , $c \to 0$

M. Niedermaier 2019

- In two dimensions, they look like

$$ds^2 = -dt^2 + \lambda^2 dx^2; \lambda > 1$$

- Since this theory is canonically equivalent to B-D theory, the constraint algebra of secondary first class constraints $(\mathcal{H}, \mathcal{H}_i)$ is like B-D theory's one.

CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- JF is canonical equivalent, via gauge-fixing of Lapse N and shifts N_i , to EF (structure of light cone preserved by JF-EF transformations).

- JF is canonical equivalent to Anti-Gravity frame (light cone structure modified by JF- Anti-Gravity transformations).

- JF cannot be equivalent to two physically inequivalent frames. Therefore, Hamiltonioan canonical transformations represent, in our opinion, a mathematical equivalence. These transformations map solutions of e.o.m into solutions of e.o.m.

CONCLUSIONS

- The transformations from the Jordan to the Einstein frames, in the extended phase space, are not Hamiltonian canonical transformations.

- Gauge-fixing the Lapse N and the Shifts N_i and implementing the Dirac's Brackets, Hamiltonian canonical transformations do exist from JF to EF.

- This very fact does not mean, necessarily, that the two frames are "physically" equivalent.

- The equivalence of the physical observables in JF and EF remains still to be studied.