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Quantum corrections to free streaming in cosmology

or

how to achieve negative thermal pressure

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OUTLINE

- Introduction
- Quantum statistical mechanics and local thermodynamic equilibrium
- The Klein-Gordon field in an expanding Universe
- Results

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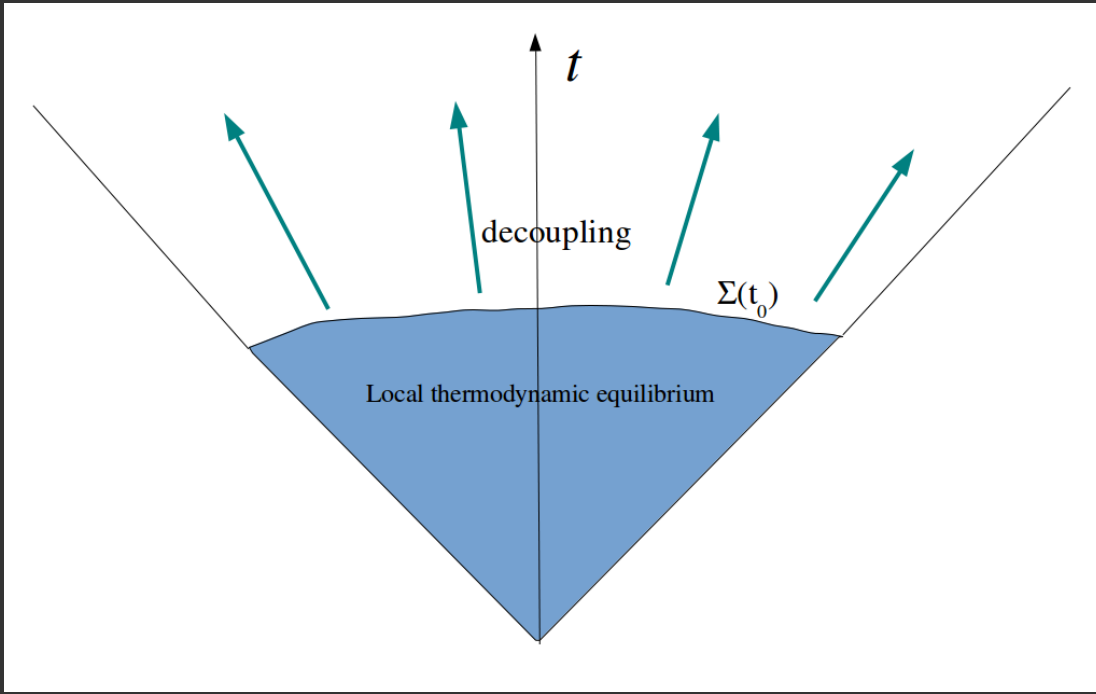
Classical free streaming

Matter at rest in the comoving coordinates at local thermodynamic equilibrium UNTIL the freeze-out or decoupling.

Classical particles decoupling: free-streaming solution of the general relativistic Boltzmann equation.



$$\varepsilon(t) = \frac{1}{(2\pi)^3 a^4(t)} \int \mathrm{d}^3\mathrm{k}\, \sqrt{\mathrm{k}^2 + m^2 a^2(t)} \frac{1}{\mathrm{e}^{\sqrt{\mathrm{k}^2+m^2}/T(t_0)} - 1}$$

$$p(t) = \frac{1}{3(2\pi)^3 a^4(t)} \int \mathrm{d}^3\mathrm{k}\, \frac{\mathrm{k}^2}{\sqrt{\mathrm{k}^2 + m^2 a^2(t)}} \frac{1}{\mathrm{e}^{\sqrt{\mathrm{k}^2+m^2}/T(t_0)} - 1}$$

k : covariant component of momentum = constant

In the long run, for a massive particle:

$$\varepsilon \simeq m \frac{N}{a^3(t)}$$

$$p \propto a(t)^{-5}$$

Quantum statistical mechanics framework:
local thermodynamic equilibrium

The state of the system at LOCAL EQUILIBRIUM is obtained by maximizing entropy with the constraint of given energy density *at each point in space*

$$S = -\mathrm{Tr}(\widehat{\rho}\log\widehat{\rho})$$

Entropy

$$\epsilon_{\mathrm{ren}}(\mathbf{x},t) = \mathrm{Tr}(\widehat{\rho}\,\widehat{\epsilon}(\mathbf{x},t)) - \langle 0|\widehat{\varepsilon}(\mathbf{x},t)|0\rangle$$

Constraints

Solution:

$$\widehat{\rho}_{\mathrm{LE}} = \frac{\exp[-\int \mathrm{d}^3\mathrm{x}\ \widehat{\varepsilon}/T(\mathbf{x},t)]}{\mathrm{Tr}\exp[-\int \mathrm{d}^3\mathrm{x}\ \widehat{\varepsilon}/T(\mathbf{x},t)]}$$

$$\Rightarrow \quad \langle\widehat{O}\rangle = \mathrm{Tr}(\widehat{\rho}_{\mathrm{LE}}\widehat{O}) - \langle 0|\widehat{O}|0\rangle$$

The density operator of the cosmological fluid at decoupling

Enforce the local equilibrium at decoupling

$$\epsilon_{\rm ren}(\mathbf{x}, t_0) = \mathrm{Tr}(\widehat{\rho}\widehat{T}^{00}(\mathbf{x}, t_0)) - \langle 0_{t_0}|\widehat{T}^{00}(\mathbf{x}, t_0)|0_{t_0}\rangle$$

$|0_{t_0}\rangle$ is the vacuum of the Hamiltonian at the time t_0

$$\widehat{H}(t_0) = \int_{\Sigma(t_0)} \mathrm{d}\Sigma_\mu \, \widehat{T}^{\mu\nu} u_\nu = a^3(t) \int_{t=t_0} \mathrm{d}^3\mathrm{x} \, \widehat{T}^{00}$$

$u_\nu = (1, 0, 0, 0)$

Matter at rest

in the Cartesian coordinates

Solution:

$$\widehat{\rho} = \frac{1}{Z} \exp[-\widehat{H}(t_0)/T(t_0)]$$

Note: for the sake of simplicity, we neglect chemical potentials

The stress-energy tensor

Vacuum-subtracted thermal expectation value of the stress-energy tensor:

$$T^{\mu\nu} = \frac{1}{Z}\mathrm{Tr}(\mathrm{e}^{-\widehat{H}(t_0)/T(t_0)}\widehat{T}^{\mu\nu}(x)) - \langle 0_{t_0}|\widehat{T}^{\mu\nu}|0_{t_0}\rangle$$

It is finite (see later on) and:

1) Comes down to normal ordering in Minkowski space-time (easy)
2) Covariantly conserved:

$$\nabla_\mu \widehat{T}^{\mu\nu} \implies \nabla_\mu T^{\mu\nu} = 0$$

because both the density operator and the vacuum are fixed

Free scalar field in FRW metric

Conformal metric $$\mathrm{d}s^2 = a^2(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2)$$ $$\eta = \int_{t_0}^{t} \frac{\mathrm{d}t}{a(t)}$$

Decoupling at η = 0

Klein-Gordon equation $$(\Box + m^2 + \xi R)\widehat{\psi} = 0$$

$$\widehat{\chi} \equiv a\widehat{\psi} \implies \chi'' + \Omega^2\chi = 0 \qquad \Omega^2 = \mathrm{k}^2 + m^2a^2 - (1-6\xi)\frac{a''}{a}$$

$$\widehat{\chi} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\; \widehat{a}(\mathbf{k})v_k(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + \widehat{a}^\dagger(\mathbf{k})v_k^*(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$v_k'' + \Omega^2 v_k = 0$$

Choice of the vacuum and eigenfunctions

The vacuum of the field is chosen to be the lowest lying state of the H(η_0)

$$\widehat{b}(\mathbf{k})|0_{\eta_0}\rangle = 0$$

$$v_k(0) = \frac{1}{\sqrt{2\omega_\xi(0,k)}} \qquad v_k'(0) = -\frac{i}{2v_k(0)} + (1-6\xi)\frac{a'(0)}{a(0)}v_k(0)$$

$$\omega_\xi(\eta,k) = \sqrt{\omega_k^2(\eta) - (1-6\xi)^2\frac{a'^2}{a^2}} = \sqrt{\mathrm{k}^2 + m^2a^2 + 6\xi(1-6\xi)\frac{a'^2}{a^2}}$$

Initial conditions for the Klein-Gordon equation!

With such eigenfunctions,

$$\widehat{H}(0) = \int \mathrm{d}^3\mathrm{k}\ \omega_\xi(0,k)\left(\widehat{a}^\dagger(\mathbf{k})\widehat{a}(\mathbf{k}) + \frac{1}{2}\right)$$

but at later times the Hamiltonan ceases to be diagonal!

Energy density and pressure

By using the definition $T^{\mu\nu} = \frac{1}{Z}\mathrm{Tr}(\mathrm{e}^{-\widehat{H}(t_0)/T(t_0)}\widehat{T}^{\mu\nu}(x)) - \langle 0_{t_0}|\widehat{T}^{\mu\nu}|0_{t_0}\rangle$

$$\varepsilon(\eta) = \frac{1}{a^4(\eta)(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\;\omega_k(\eta)K_k(\eta)n_B\left(\frac{\omega_\xi(0,k)}{T(0)}\right),$$

$$p(\eta) = \frac{1}{a^4(\eta)(2\pi)^3}\int \mathrm{d}^3\mathrm{k}\;\omega_k(\eta)\Gamma_k(\eta)n_B\left(\frac{\omega_\xi(0,k)}{T(0)}\right).$$

$$n_B\left(\frac{\omega_\xi(0,k)}{T(0)}\right) \equiv \frac{1}{\exp\left[\omega_\xi(0,k)/T(0)\right]-1},$$

$$K_k(\eta) = \frac{1}{\omega_k}\left[|v_k'|^2 + \omega_k^2|v_k|^2 - 2(1-6\xi)\frac{a'}{a}\mathrm{Re}(v_k' v_k^*)\right]$$

$$\Gamma_k(\eta) = \frac{1}{\omega_k(\eta)}\left[(1-4\xi)\,|v_k'(\eta)|^2 + \frac{1}{3}\gamma_k(\eta)\,|v_k(\eta)|^2 - 2(1-6\xi)\frac{a'(\eta)}{a(\eta)}\mathrm{Re}(v_k'(\eta)v_k^*(\eta))\right]$$

$$\omega_k^2(\eta) = \mathrm{k}^2 + m^2a^2 + (1-6\xi)\frac{a'^2}{a^2}$$

$$\gamma_k(\eta) = (12\xi-1)\,\mathrm{k}^2 + 3(4\xi-1)m^2a^2(\eta) + 3(1-6\xi)\frac{a'^2(\eta)}{a^2(\eta)} - 12\xi(1-6\xi)\frac{a''(\eta)}{a(\eta)}$$

Quantum corrections to energy density and pressure

For $\xi = 0$ (minimal coupling) or $\xi=1/6$ (conformal coupling)

$$\Delta\varepsilon(\eta) = \frac{1}{a^4(\eta)(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left[\omega_k(\eta) K_k(\eta) - \sqrt{\mathrm{k}^2 + m^2 a^2(\eta)}\right] n_B \left(\frac{\omega_k(0)}{T(0)}\right),$$

$$\Delta p(\eta) = \frac{1}{a^4(\eta)(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \left[\omega_k(\eta) \Gamma_k(\eta) - \frac{\mathrm{k}^2}{3\sqrt{\mathrm{k}^2 + m^2 a^2(\eta)}}\right] n_B \left(\frac{\omega_k(0)}{T(0)}\right).$$

$$K_k^2(\eta) - |\Lambda_k(\eta)|^2 = \frac{\omega_\xi^2(\eta, k)}{\omega_k^2(\eta)} \quad \Longrightarrow \quad \Delta\varepsilon > 0$$

At the decoupling they are precisely zero, but they get non-vanishing thereafter

Can they become significant in the long run?

This depends on the evolution of the eigenfunctions, governed by the equation

$$v_k'' + \Omega^2 v_k = 0$$

Adiabatic solution

Slowly expanding Universe: $\dfrac{\Omega'}{\Omega^2} \ll 1$

$$v_k'' + \Omega^2 v_k = 0 \qquad\qquad \Omega^2 = \mathrm{k}^2 + m^2 a^2 - (1-6\xi)\frac{a''}{a}$$

$$v_k \simeq A_k \frac{\mathrm{e}^{-i\int \Omega(\eta)\mathrm{d}\eta}}{\sqrt{2\Omega}} + B_k \frac{\mathrm{e}^{i\int \Omega(\eta)\mathrm{d}\eta}}{\sqrt{2\Omega}}$$

Integrands of ε

$$\omega_k + 2\omega_k \left|B_k\right|^2$$

p

$$\frac{\mathrm{k}^2}{3\omega_k} + \frac{2\mathrm{k}^2}{3\omega_k}\left|B_k\right|^2 - \frac{(1-8\xi)\,\omega_k^2 + m^2a^2 + \mathrm{k}^2/3}{\omega_k}\mathrm{Re}\left[A_k B_k^* e^{-2i\int_k \omega_k}\right].$$

They are the energy density and pressure with the addition of a “cosmological particle production” term. Correction size depends on *B/A*

Non adiabatic regime $(\xi = 0)$

$v_k'' + \Omega^2 v_k = 0$ Back to cosmological time $u_k = v_k / a$

$$\ddot{u}_k + 3\frac{\dot{a}}{a}\dot{u}_k + \left(\frac{k^2}{a^2} + m^2\right) u_k = 0$$

If $\frac{\dot{a}}{a} \equiv H \gg m$ overdamped oscillations and $u_k \approx \exp[-(m^2/3H)t]$

The pressure goes negative because $\dot{u}_k \ll m u_k$

$$p = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathrm{k} \; n_B\left(\frac{\omega_k(0)}{T(0)}\right) \left[|\dot{u}_k|^2 - \left(\frac{k^2}{3a^2} + m^2\right)|u_k|^2\right]$$

In general, the pressure can go negative in the long run if *m*=0 or with very low *m* and an accelerated Universe

Solutions with fixed background

- De Sitter space-time $\dfrac{\dot{a}}{a} = H = \mathrm{const}$

$$v_k(y) = A_k\sqrt{\frac{\pi}{4H}}\sqrt{y}\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}y}{H}\right) + \sqrt{\frac{\pi}{4H}}\sqrt{y}\mathrm{H}_\nu^{(2)}\left(\frac{\mathrm{k}y}{H}\right),$$

$$\nu = \frac{1}{2}\sqrt{9 - 48\xi - \frac{4m^2}{H^2}}, \qquad y = 1 - H\eta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\eta} = -H,$$

$$A_k = \sqrt{\frac{4H}{\pi}}\frac{v_k(1)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{H}\right)} + \sqrt{\frac{4H}{\pi}}\frac{\pi\mathrm{H}_\nu^{(2)}\left(\frac{\mathrm{k}}{H}\right)}{8Hv_k(1)}\left[1 + iHv_k^2(1)\left(3 - 12\xi + 2\nu - 2\frac{\mathrm{k}}{H}\frac{\mathrm{H}_{\nu+1}^{(1)}\left(\frac{\mathrm{k}}{H}\right)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{H}\right)}\right)\right],$$

$$B_k = -\sqrt{\frac{4H}{\pi}}\frac{\pi\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{H}\right)}{8Hv_k(1)}\left[1 + iHv_k^2(1)\left(3 - 12\xi + 2\nu - 2\frac{\mathrm{k}}{H}\frac{\mathrm{H}_{\nu+1}^{(1)}\left(\frac{\mathrm{k}}{H}\right)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{H}\right)}\right)\right].$$

- Generally accelerated-decelerated space-time $a(t) = (t/t_0)^\sigma$ with m=0

$$v_k(y) = A_k\sqrt{\frac{\pi}{4H}}\sqrt{y}\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}y}{h}\right) + B_k\sqrt{\frac{\pi}{4H}}\sqrt{y}\mathrm{H}_\nu^{(2)}\left(\frac{\mathrm{k}y}{h}\right),$$

$$h = a'/a = \frac{\sigma - 1}{t_0}$$

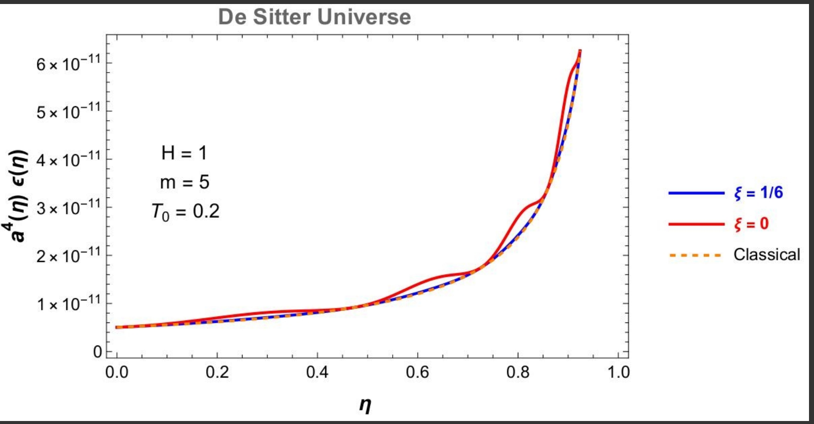
$$\nu = \frac{1}{2}\sqrt{\frac{4\sigma(2\sigma-1)(1-6\xi)}{(1-\sigma)^2} + 1}$$

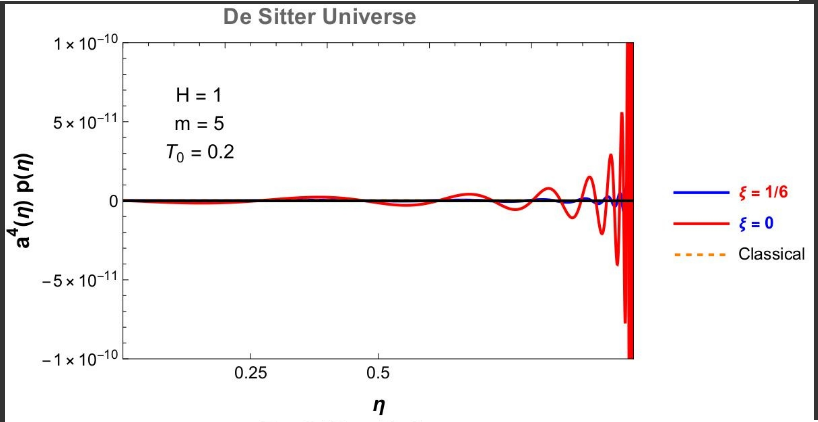
$$\begin{cases} A_k = \sqrt{\frac{4h}{\pi}}\frac{v_k(1)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{h}\right)} + \sqrt{\frac{4h}{\pi}}\frac{\pi\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{h}\right)}{8hv_k(1)}\left[1 + ihv_k^2(1)\left(\frac{\sigma(3-12\xi)-1}{\sigma-1} + 2\nu - 2\frac{\mathrm{k}}{h}\frac{\mathrm{H}_{\nu+1}^{(1)}\left(\frac{\mathrm{k}}{h}\right)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{h}\right)}\right)\right], \\ B_k = -\sqrt{\frac{4h}{\pi}}\frac{\pi\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{h}\right)}{8hv_k(1)}\left[1 + ihv_k^2(1)\left(\frac{\sigma(3-12\xi)-1}{\sigma-1} + 2\nu - 2\frac{\mathrm{k}}{h}\frac{\mathrm{H}_{\nu+1}^{(1)}\left(\frac{\mathrm{k}}{h}\right)}{\mathrm{H}_\nu^{(1)}\left(\frac{\mathrm{k}}{h}\right)}\right)\right], \end{cases}$$

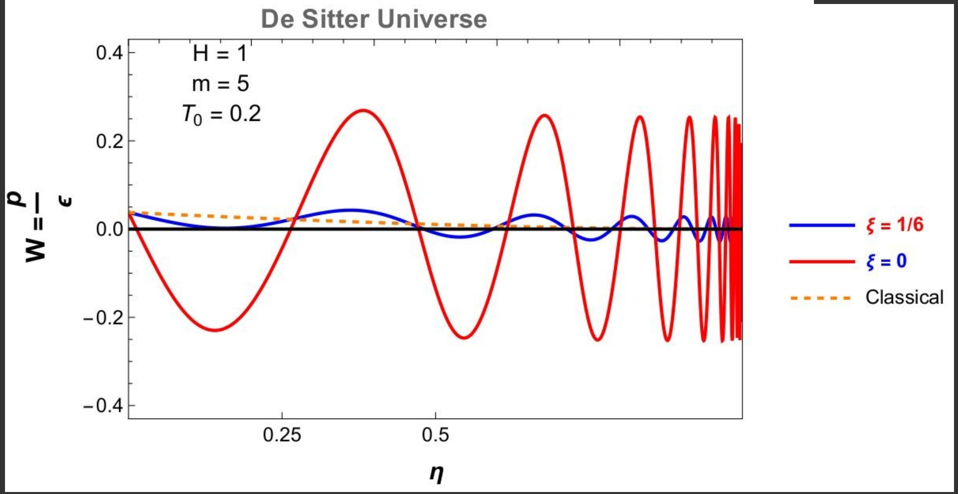
$$y = 1 - h\eta,$$

Results: De Sitter space-time

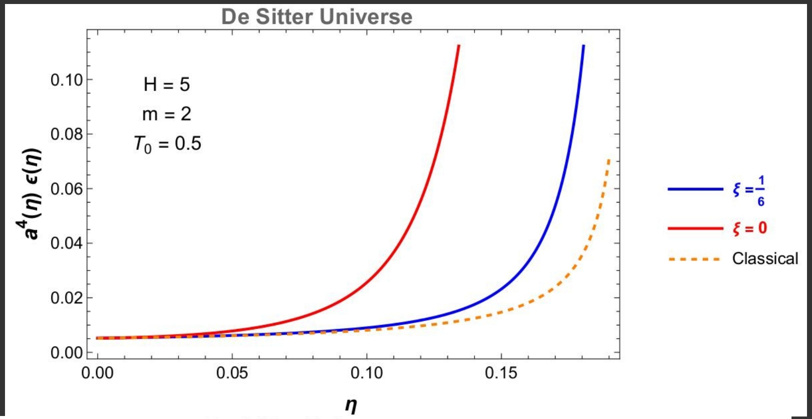
Adiabatic

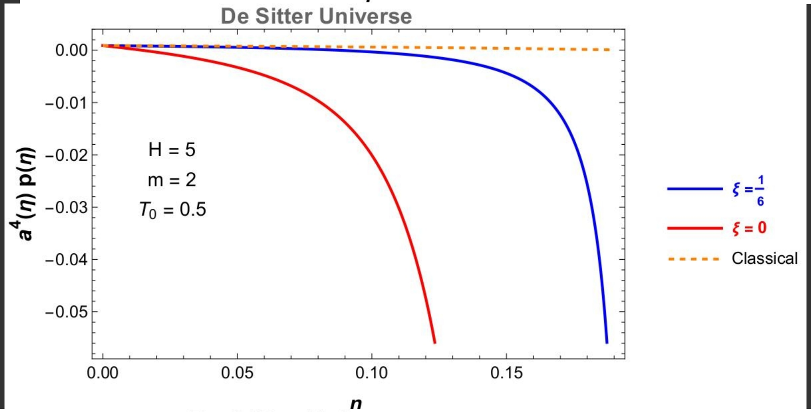


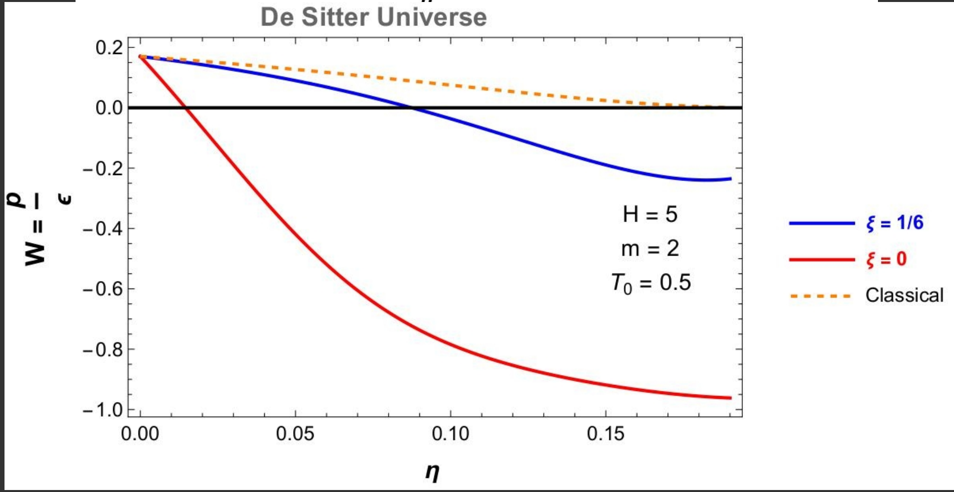




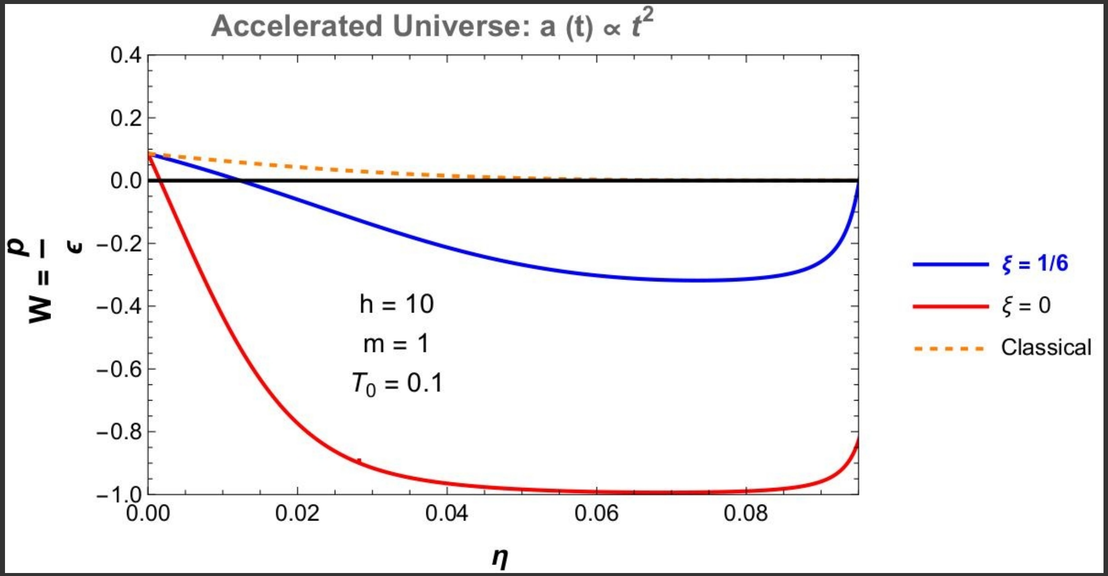
Non-adiabatic





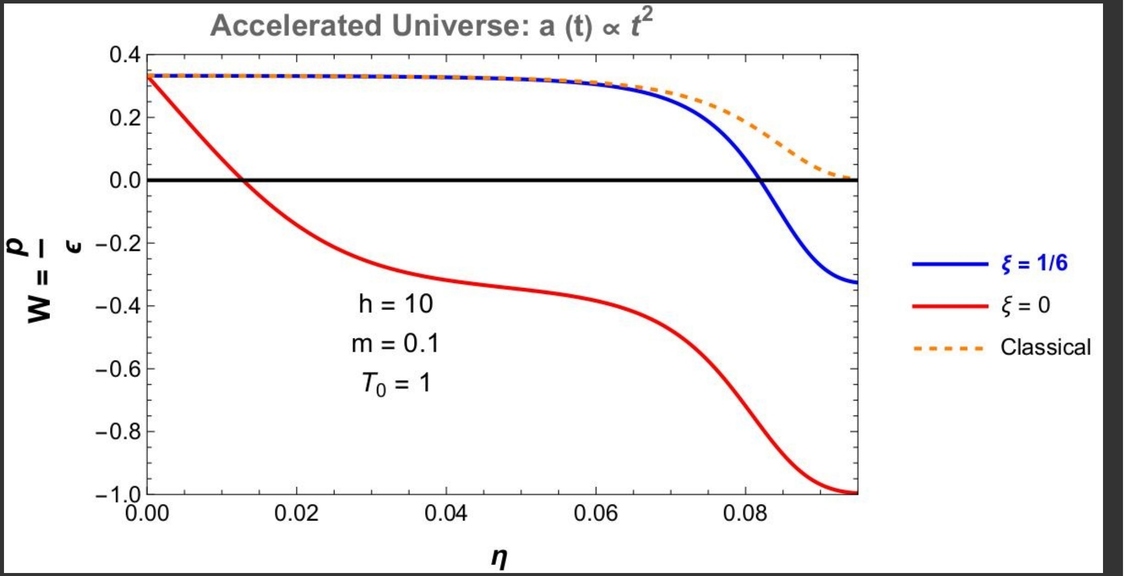


Accelerated Universe $a(t) = (t/t_0)^2$



NOTE: $h = a'/a$ at the decoupling

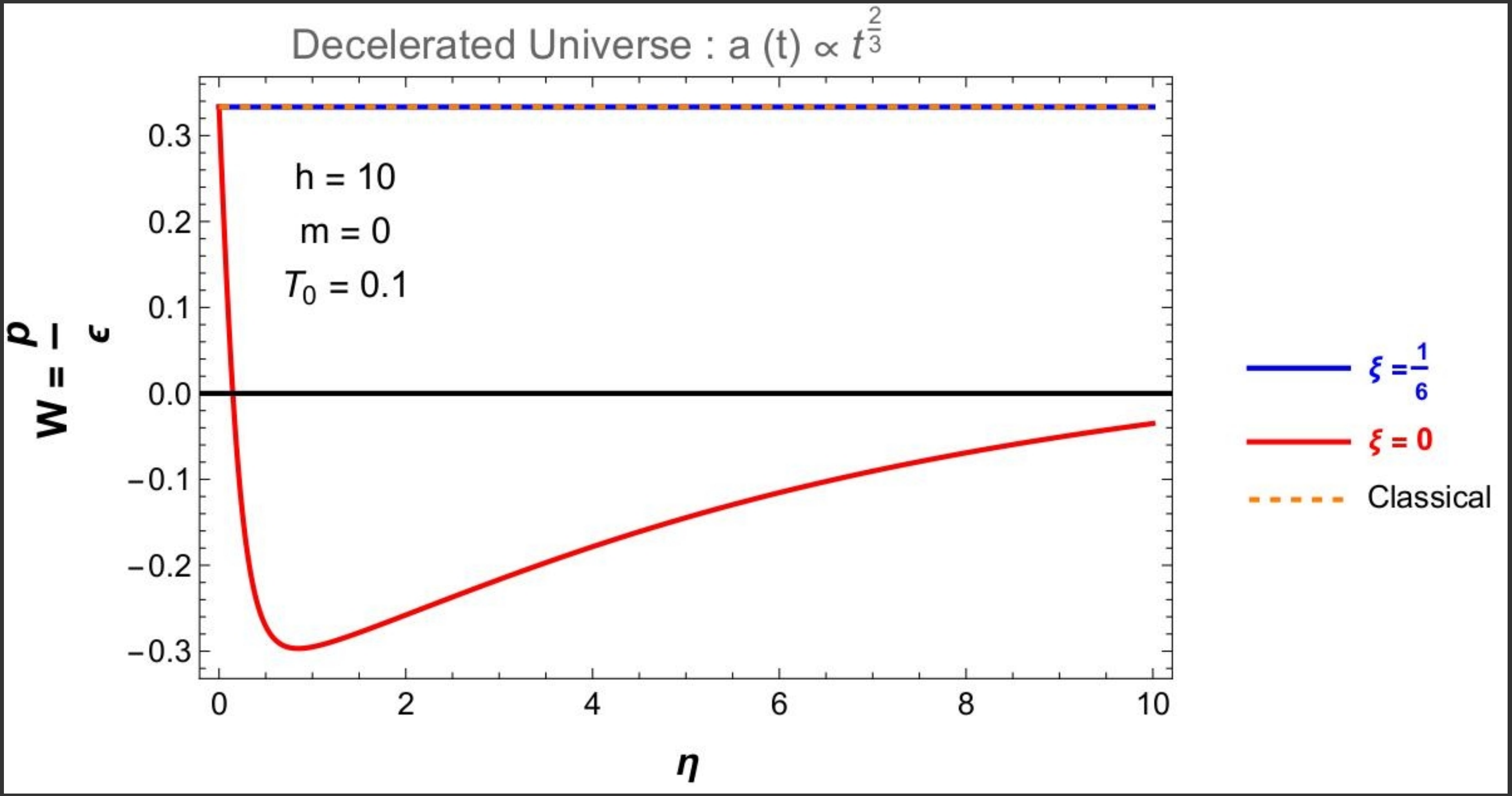
$H = h$ at the decoupling



Decelerated Universe $a(t) = (t/t_0)^{2/3}$

NOTE: $h = a'/a$ at the decoupling

$H = h$ at the decoupling



Conclusions and outlook

- Quantum corrections to free-streaming in the cosmological metric from quantum statistical mechanics and Klein-Gordon equation in curved background

- Corrections to energy density and pressure can be relevant depending on the relative magnitude of the expansion rate, mass and decoupling temperature.

We have studied selected exact solutions with given scale function *a(t)*

- Energy density is generally enhanced w.r.t. the classical value

- Pressure, intriguingly, can become negative if the expansion rate exceeds the mass of the minimally coupled field

OUTLOOK

- Study of the coupled differential equations of the field and the metric

Quantum statistical mechanics framework

The state of the system at global equilibrium is obtained by maximizing entropy with the constraint of given energy

$$S = -\mathrm{Tr}(\widehat{\rho} \log \widehat{\rho})$$ Entropy

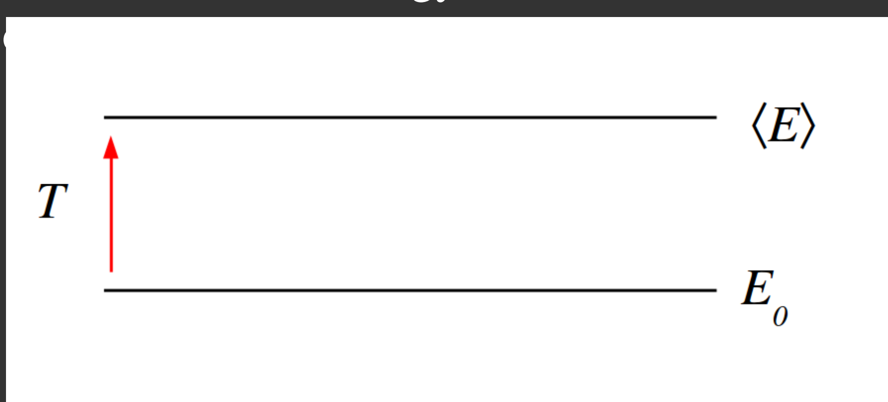
$$E_{\mathrm{ren}} = \mathrm{Tr}(\widehat{\rho}\, \widehat{H}) - \langle 0|\widehat{H}|0\rangle$$ Constraint

$1/T$ is
the Lagrange multiplier

Function to be maximized $$-\mathrm{Tr}(\widehat{\rho} \log \widehat{\rho}) + \frac{1}{T}\left(\mathrm{Tr}(\widehat{\rho}\widehat{T}) - \langle 0|\widehat{H}|0\rangle - E_{\mathrm{ren}}\right)$$

Solution: $$\widehat{\rho} = \frac{\exp[-\widehat{H}/T(E_{\mathrm{ren}})]}{\mathrm{Tr}\exp[-\widehat{H}/T(E_{\mathrm{ren}})]}$$

Note that temperature – obtained by solving the constraint - is a function of the renormalized, true value of the energy



Finite T > 0 quantifies the degree of excitation with respect to the vacuum,
i.e. the lowest lying state of the Hamiltonian

Eigenfunctions and vacuum ambiguity

Klein-Gordon product $$(\phi_1, \phi_2) = i \int \mathrm{d}\Sigma_\mu \, (\phi_1 \nabla^\mu \phi_2^* - \phi_2^* \nabla^\mu \phi_1)$$

$$W[v, v^*] = v v'^* - v^* v' = i \quad \Longleftrightarrow \quad [\widehat{a}(\mathbf{k}), \widehat{a}^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k} - \mathbf{k}')$$

Example in Minkowski: $v = \dfrac{1}{\sqrt{2\omega}} \mathrm{e}^{-i\omega\eta} \qquad \omega = \sqrt{\mathrm{k}^2 + m^2}$

Vacuum ambiguity and Bogoliubov transformations

New set of eigenfunctions: $$w_k = A_k v_k + B_k v_k^*$$

$$W[w_k, w_k^*] = i \implies A_k^2 - B_k^2 = 1$$

Corresponding new set of creation/annihilation operators

$$\widehat{b}(\mathbf{k}) = A_k^* \widehat{a}(\mathbf{k}) - B_k^* \widehat{a}^\dagger(-\mathbf{k})$$
$$\widehat{a}(\mathbf{k}) = A_k \widehat{b}(\mathbf{k}) + B_k^* \widehat{b}^\dagger(-\mathbf{k})$$

$$|0\rangle_a \neq |0\rangle_b$$

Stress-energy tensor of the KG field

$$\widehat{T}_{\mu\nu} = \nabla_\mu \widehat{\psi} \nabla_\nu \widehat{\psi} - \frac{1}{2} g_{\mu\nu} \left(\nabla \widehat{\psi} \cdot \nabla \widehat{\psi} - \frac{1}{2} m^2 \widehat{\psi}^2 \right) + \xi \left(G_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \widehat{\psi}^2$$

$$\widehat{T}_{00} = \frac{1}{2a^4} \left[\widehat{\chi}'^2 + |\mathrm{grad} \widehat{\chi}|^2 + m^2 a^2 \widehat{\chi}^2 + (1 - 6\xi) \left(\frac{a'^2}{a^2} \widehat{\chi}^2 - 2 \frac{a'}{a} \widehat{\chi}' \widehat{\chi} \right) \right]$$

00- Cartesian component

The Hamiltonian at the conformal time h turns out to be, by using the field expansion:

$$\widehat{H}(\eta) = \frac{1}{2a(\eta)} \int \mathrm{d}^3\mathrm{k}\, \omega_k(\eta) \left[K_k(\eta) \left(\widehat{a}^\dagger(\mathbf{k}) \widehat{a}(\mathbf{k}) + \widehat{a}(\mathbf{k}) \widehat{a}^\dagger(\mathbf{k}) \right) + \Lambda_k(\eta) \widehat{a}(\mathbf{k}) \widehat{a}(-\mathbf{k}) + \Lambda_k^*(\eta) \widehat{a}^\dagger(-\mathbf{k}) \widehat{a}^\dagger(\mathbf{k}) \right]$$

$$\omega_k^2(\eta) = \mathrm{k}^2 + m^2 a^2 + (1 - 6\xi) \frac{a'^2}{a^2}$$

$$K_k(\eta) = \frac{1}{\omega_k} \left[|v_k'|^2 + \omega_k^2 |v_k|^2 - 2(1 - 6\xi) \frac{a'}{a} \mathrm{Re}(v_k' v_k^*) \right]$$

$$\Lambda_k(\eta) = \frac{1}{\omega_k} \left[v_k'^2 + \omega_k^2 v_k^2 - 2(1 - 6\xi) \frac{a'}{a} v_k' v_k \right]$$

Expectation values

They can be calculated easily in the b basis:

$$\langle \widehat{b}^\dagger(\mathbf{k})\widehat{b}(\mathbf{k}')\rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\widehat{H}(\eta_0)/T(\eta_0)}\widehat{b}^\dagger(\mathbf{k})\widehat{b}(\mathbf{k}'))$$

$$= \frac{\delta^3(\mathbf{k}-\mathbf{k}')}{\exp[-\sqrt{\mathrm{k}^2+m^2}/T(\eta_0)]-1} \equiv n_B\left(\frac{\omega_c(\eta_0)}{T(\eta_0)}\right)\delta^3(\mathbf{k}-\mathbf{k}')$$

$$\langle \widehat{b}(\mathbf{k})\widehat{b}(\mathbf{k}')\rangle = 0$$

Can be used to calculate the renormalized stress-energy tensor:

$$\langle \widehat{T}^{\mu\nu}\rangle_{\mathrm{ren}} = \frac{1}{Z}\mathrm{Tr}(\mathrm{e}^{-\widehat{H}(\eta_0)/T(\eta_0)}\widehat{T}^{\mu\nu}(x)) - \langle 0_{\eta_0}|\widehat{T}^{\mu\nu}|0_{\eta_0}\rangle$$

Minkowski space-time

For a free field, this is defined

$$\langle \widehat{T}^{\mu\nu} \rangle_{\rm ren} \equiv {\rm Tr}(\widehat{\rho} : \widehat{T}^{\mu\nu} :) = {\rm Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}) - \langle 0|\widehat{T}^{\mu\nu}|0\rangle$$

For the familiar

$$\widehat{\rho} = \frac{1}{Z} \exp -\widehat{H}/T$$

This is expressed as:

$$\langle \widehat{T}^{\mu\nu} \rangle_{\rm ren} = {\rm Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}) - \lim_{T\to 0} {\rm Tr}(\widehat{\rho}\widehat{T}^{\mu\nu})$$

Which can be used for interacting theories as well