On the dynamics of perturbed Black Holes

Michele Lenzi

Institut de Ciències de l'Espai (ICE-CSIC, IEEC), Barcelona

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M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021), Phys. Rev. D **104**, 124068 (2021), Phys. Rev. D **107**, 044010 (2023), Phys. Rev. D **107**, 084039 (2023)

Outline

Full landscape of master equations for non-rotating BHs

Decoupling Einstein equations

- Perturbed Einstein equations at linear order

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad \widehat{G}_{\mu\nu} = 0\,, \quad \delta G_{\mu\nu} = 0$$

- Metric splitting reflecting spherical symmetry

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -f(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/f(r) \\ \Omega_{AB}\mathrm{d}\Theta^A\mathrm{d}\Theta^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \end{array}$$

- Harmonics expansion $\quad h_{\mu\nu} = \sum_{\ell,m} h_{\mu\nu}^{\ell m,\mathrm{odd}} + h_{\mu\nu}^{\ell m,\mathrm{even}}$

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- Harmonics expansion $\quad h_{\mu\nu} = \sum_{\ell,m} h_{\mu\nu}^{\ell m,\text{odd}} + h_{\mu\nu}^{\ell m,\text{even}}$
- The master equations

$$\delta G_{\mu\nu} = 0 \quad \longrightarrow \quad \boxed{\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V^{\ell}_{\text{even/odd}}\right)\Psi^{\ell m}_{\text{even/odd}} = 0}$$

Known master equations

Odd parity

$$
\begin{aligned}
\Psi_{\mathrm{RW}} &= \frac{r^a}{r}\tilde{h}_a \\
\Psi_{\mathrm{CPM}} &= \frac{2r}{(\ell-1)(\ell+2)}\varepsilon^{ab}\left(\tilde{h}_{b:a} - \frac{2}{r}r_a\tilde{h}_b\right) \\
V_{\mathrm{RW}} &= f(r)\left(\frac{\ell(\ell+1)}{r^2} - \frac{3r_s}{r^3}\right)
\end{aligned}
$$

T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063–1069 (1957), C. T. Cunningham et al., Astrophys. J. **224**, 643–667 (1978)

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$$V_{\mathrm{RW}} = f(r)\left(\frac{\ell(\ell+1)}{r^2} - \frac{3r_s}{r^3}\right)$$

Even parity

$$\Psi_{\mathrm{ZM}} = \frac{2r}{\ell(\ell+1)}\left\{\tilde{K} + \frac{2}{\lambda}\left(r^a r^b \tilde{h}_{ab} - r r^a \tilde{K}_{:a}\right)\right\}$$

$$V_{\mathrm{Z}} = \frac{f(r)}{\lambda^2}\left[\frac{(\ell-1)^2(\ell+2)^2}{r^2}\left(\ell(\ell+1) + \frac{3r_s}{r}\right) + \frac{9r_s^2}{r^4}\left((\ell-1)(\ell+2) + \frac{r_s}{r}\right)\right]$$

F. J. Zerilli, Phys. Rev. D **2**, 2141–2160 (1970), V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974)

Assumptions

What are all the possible master equations that one can obtain for the vacuum perturbations of a Schwarzschild BH?

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

Assumptions

What are all the possible master equations that one can obtain for the vacuum perturbations of a Schwarzschild BH?

1. Linear in the metric perturbations and first-order derivatives

$$\begin{aligned} \Psi_{\text{odd}}^{\ell m} &= C_0^\ell\, h_0^{\ell m} + C_1^\ell\, h_1^{\ell m} + C_2^\ell\, h_2^{\ell m} \\ &+ C_3^\ell\, \dot{h}_0^{\ell m} + C_4^\ell\, h_0'^{\ell m} + C_5^\ell\, \dot{h}_1^{\ell m} \\ &+ C_6^\ell\, h_1'^{\ell m} + C_7^\ell\, \dot{h}_2^{\ell m} + C_8^\ell\, h_2'^{\ell m} \end{aligned}$$

2. Time independent coefficients

$$C_i^\ell = C_i^\ell(r)$$

3. Arbitrary perturbative gauge

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

The standard branch

- Standard branch potentials

$$
{}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}} = \begin{cases} V_\ell^{\mathrm{RW}} & \text{odd parity} \\ V_\ell^{\mathrm{Z}} & \text{even parity} \end{cases}
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- Most general master function

$$ {}_{\rm S}\Psi^{\rm odd/even} = \begin{cases} \mathcal{C}_1 \Psi^{\rm CPM} + \mathcal{C}_2 \Psi^{\rm RW} & \text{odd parity} \\ \mathcal{C}_1 \Psi^{\rm ZM} + \mathcal{C}_2 \boxed{\Psi^{\rm NE}} & \text{even parity} \end{cases} $$

$$ \Psi^{\rm NE}(t,r) = \frac{1}{\lambda(r)} t^a \left(r \tilde{K}_{:a} - \tilde{h}_{ab} r^b \right) $$

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- Time derivative relation

$$
t^a \Psi^{\mathrm{CPM}}_{,a} = 2\,\Psi^{\mathrm{RW}}\,, \quad t^a \Psi^{\mathrm{ZM}}_{,a} = 2\,\Psi^{\mathrm{NE}}
$$

The Darboux branch

- Family of potentials ${}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}$ satisfying

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{V_{\ell,x}^{\mathrm{RW/Z}}}{\delta V}\right)_{,x} - \delta V = 0\,,$$

with $\delta V = {}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}} - V_\ell^{\mathrm{RW/Z}}$.

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- Most general (potential dependent) master function

$${}_{\mathrm{D}}\Psi^{\mathrm{odd/even}} = \begin{cases} \mathcal{C}_1 \Psi^{\mathrm{CPM}} + \mathcal{C}_2\left(\Sigma^{\mathrm{odd}}\,\Psi^{\mathrm{CPM}} + \Phi^{\mathrm{odd}}\right) & \text{odd parity} \\ \mathcal{C}_1 \Psi^{\mathrm{ZM}} + \mathcal{C}_2\left(\Sigma^{\mathrm{even}}\,\Psi^{\mathrm{ZM}} + \Phi^{\mathrm{even}}\right) & \text{even parity} \end{cases}$$

Darboux covariance in perturbed Schwarzschild BH

Darboux transformation

- Darboux transformation between (v, Φ) and (V, Ψ)

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = 0 \longrightarrow \begin{cases} \Psi = \Phi_{,x} + g\,\Phi \\ V = v + 2\,g_{,x} \\ g_{,x} - g^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0$$

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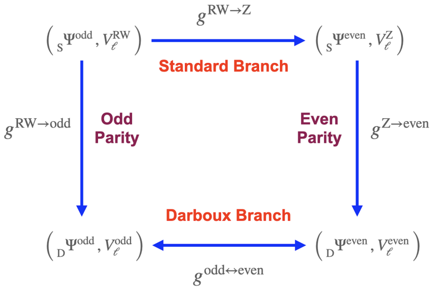
- Darboux covariance of perturbations of spherically-symmetric BHs



M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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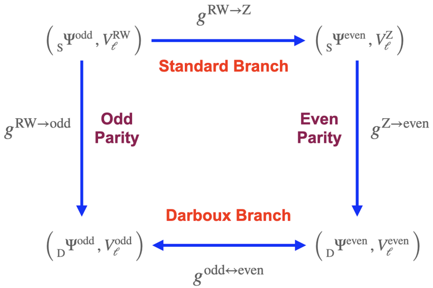

Isospectral symmetry

$$\Psi = e^{ikt}\psi$$

$$\phi_{,xx} - v\phi = -k^2\phi$$
$$\psi_{,xx} - V\psi = -k^2\psi$$

Korteweg-de Vries isospectral deformations

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$$V_{,\tau} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation

C. S. Gardner et al., Phys. Rev. Lett. **19**, 1095–1097 (1967)

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- Darboux transformation + inverse scattering solves the KdV equation
- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\tau, x) \\ \psi(x) \to \psi(\tau, x) \\ k \to k(\tau) \end{cases}$$

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$I_n[V] = \int_{-\infty}^{\infty} dx\, P_n(V, V_{,x}, V_{,xx}, \ldots)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

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$$I_n[V] = \int_{-\infty}^{\infty} dx\, P_n(V, V_{,x}, V_{,xx}, \ldots) \quad \longrightarrow \quad \boxed{I_n[V] = I_n[V_{\mathrm{RW}}]}$$

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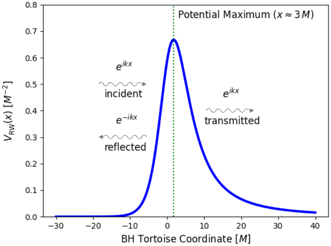
Greybody factors from KdV integrals: moment problem methods

BH scattering

$$\psi(x,k,\tau) = \begin{cases} a(k,\tau)e^{ikx} + b(k,\tau)e^{-ikx} \\ e^{ikx} \end{cases}$$



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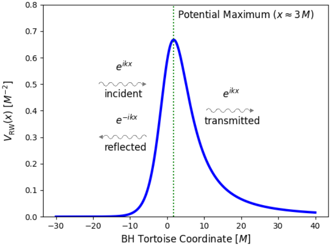

- Bogoliubov coefficients completely determine the physics (greybody factors and QNMs)
  - Greybody factors

$$T(k,\tau) = |a(k,\tau)|^{-2}\ , \quad R(k,\tau) = \left|\frac{b(k,\tau)}{a(k,\tau)}\right|^2$$

  - QNMs: k_i such that $a(k_i,\tau) = 0$

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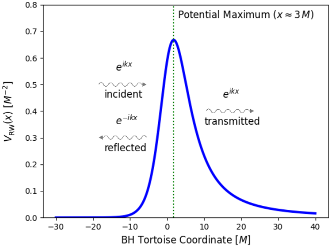

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  - QNMs: k_i such that $a(k_i,\tau) = 0$
- Greybody factors and QNMs are conserved by DT and KdV deformations

BH greybody factors from KdV integrals

- Trace identities: a set of integral equations that relate the KdV integrals to the greybody factors

$$(-1)^{n+1}\frac{I_{2n+1}}{2^{2n+1}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, k^{2n} \ln T(k)$$

BH greybody factors from KdV integrals

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A moment problem

The greybody factors in BH scattering processes are uniquely determined by the KdV integrals of the BH potential via a (Hamburger) moment problem

$$\mu_{2n} = \int_{-\infty}^{\infty} dk\, k^{2n} p(k)$$

where

$$\mu_{2n} = (-1)^n\frac{I_{2n+1}}{2^{2n+1}}\,, \quad p(k) = -\frac{\ln T(k)}{2\pi}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023)

Moment problem

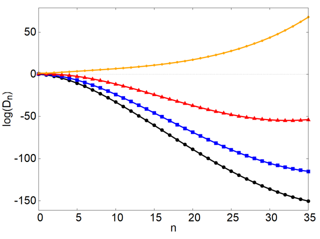
$$\mu_n = \int_{\mathcal{I}} dx\, x^n\, p(x) \quad n = 0, 1, 2, \ldots$$

- Existence: Is there a function $p(x)$ on $\mathcal{I}$ whose moments are given by $\{\mu_n\}$?
- Uniqueness: Do the moments $\{\mu_n\}$ determine uniquely a distribution $p(x)$ on $\mathcal{I}$?
- Solution: How can we construct all such probability distributions?

Moment problem: Existence and Uniqueness

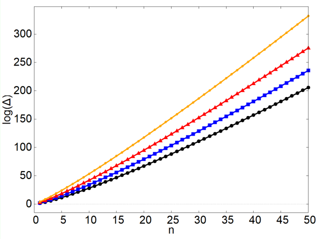
Existence

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} > 0$$



Uniqueness

$$\Delta(n) = C^n(2n)! - \hat{\mu}_{2n} > 0$$



Moment problem: Solution

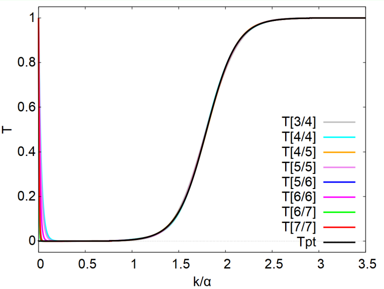
Solution through Padé approximants

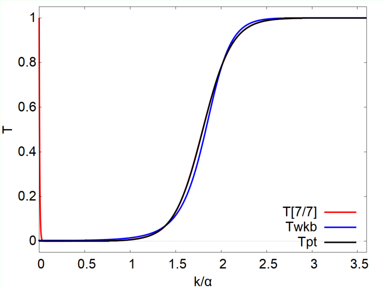
$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \quad \lambda_i = \lambda_i\left[\{I_n\}\right] \quad t_i = t_i\left[\{I_n\}\right]$$

Moment problem: Solution

Solution through Padé approximants: Pöschl-Teller

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \qquad \lambda_i = \lambda_i\,[\{I_n\}] \qquad t_i = t_i\,[\{I_n\}]$$

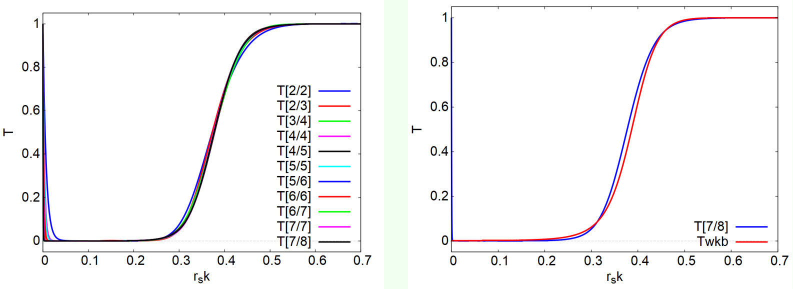




Moment problem: Solution

Solution through Padé approximants: Regge-Wheeler

$$T(k) \simeq \exp\left(-2\,\pi\,\sigma\,k\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \quad \lambda_i = \lambda_i\left[\{I_n\}\right] \quad t_i = t_i\left[\{I_n\}\right]$$



Conclusions and Outlooks

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1. Extension to higher perturbative orders
2. Moment problem to explore alternative theories, BH mimickers...
3. QNMs from KdV integrals: peering into QNMs instability?
4. Study of Kerr BHs perturbations