Grand canonical ensemble of a d-dimensional Reissner-Nordström black hole space inside a cavity

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Motivation

Gibbons and Hawking[1] obtained thermodynamics of BH by constructing canonical and grand canonical ensemble through the path integral approach in the zero loop approximation (but i.e. canonical ensemble of Schwarzschild was unstable).

York[2] assumed finite cavity, one Schwarzschild solution was stable!

Braden et al[3] analyzed the grand canonical ensemble of a charged black hole.

André and Lemos[4] generalized York's work for *d* dimensions.

[1] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[2] J. W. York, Phys. Rev. D **33**, 2092 (1986)
[3] H. W. Braden, J. D. Brown, B. F. Whiting and J. W. York, Phys. Rev. D **42**, 3376 (1990).
[4] R. André and J. P. S. Lemos, Phys. Rev. D **103**, 064069 (2021)

Partition function through Euclidean path integral approach

The partition function by the Euclidean path integral approach is $Z[\beta,\phi] = \int DADg \, \mathrm{e}^{-I}$.

One transforms the Lorentzian spacetime to Euclidean space through $t \to -i\tau$, where τ is periodic.

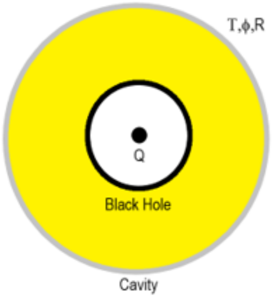
Outer boundary is spherical with radius R. The inverse temperature β is the Euclidean time length at this boundary and ϕ is the electric potential. Regularity conditions at inner "boundary".

Zero Loop approximation:
Assume only spherical symmetric metrics.
Apply GR and Maxwell constraints to get reduced action I_*, then find its minimum (black hole solutions).

Action, geometry and boundary conditions

The Euclidean action is

$$I_E = -\int_{\mathscr{M}} \left(\frac{R}{16\pi} - \frac{F_{ab}F^{ab}}{4} \right) \sqrt{g}\, d^d x$$
$$-\frac{1}{8\pi} \int_{\mathscr{B}} (K - K_0) \sqrt{\gamma}\, d^{d-1} x \,,$$
$$F_{ab} = \partial_a A_b - \partial_b A_a \,.$$



We impose spherical symmetric metric

$$ds^2 = b^2 d\tau^2 + \alpha^2 dy^2 + r^2 d\Omega^2 \,,\; 0 \leq \tau \leq 2\pi \,,\; 0 \leq y \leq 1 \,.$$

Boundary (regularity) conditions at $y = 0$: $b(0) = 0$, $b' \alpha^{-1}\big|_0 = 1$, $r' \alpha^{-1}\big|_0 = 0$, $A_\tau(0) = 0$, $r(0) = r_+$.

Boundary conditions at $y = 1$: $\beta = 2\pi b(1)$ and $\beta \phi = 2\pi i A_\tau(1)$.

Reduced action

We impose the Gauss constraint, the Hamiltonian and momentum constraints. Then, the reduced action becomes

$$I_* = \frac{(d-2)\Omega R^{d-3}\beta}{8\pi}\left(1-\sqrt{f[R;r_+,q]}\right) - q\beta\phi - \frac{\Omega r_+^{d-2}}{4}\,.$$

$$f = 1 - \frac{r_+^{d-3}}{r^{d-3}} - \frac{\lambda q^2}{r_+^{d-3} r^{d-3}} + \frac{\lambda q^2}{r^{2d-6}}\ ,\ \ \lambda = \frac{8\pi}{(d-2)(d-3)\Omega^2}\,.$$

The stationary points $r_+[\beta,\phi,R]$ and $q[\beta,\phi,R]$, where $dI_* = 0$, are found by

$$\beta = \frac{1}{T} = \frac{4\pi}{d-3}\frac{r_+^{2d-5}}{r_+^{2d-6} - \lambda q^2}\sqrt{f}\,,$$

$$\phi = \frac{q}{(d-3)\Omega\sqrt{f}}\left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}}\right)\,.$$

Solutions of ensemble and stability

There are up to 2 solutions $r_{+1} < r_{+2}$.
We define:

$x = \frac{r_+}{R}$, $y = \frac{\lambda q^2}{R^{2d-6}}$, $\lambda = \frac{8\pi}{(d-2)(d-3)\Omega^2}$,

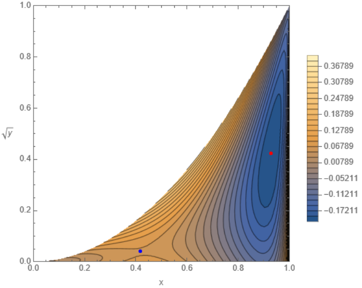
$\Phi = (d-3)\Omega\sqrt{\lambda}\phi$,

$$\pi R T_0 = \frac{(d-3)^{\frac{1}{2}}(d-1)^{\frac{d-1}{2d-6}}(1-\Phi^2)^{\frac{d-2}{d-3}}}{2^{\frac{2d-5}{d-3}}} .$$

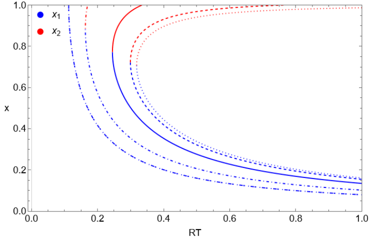
Solution r_{+2} (stable) exists in:
$RT_0 < RT < (d-3)\frac{1-\Phi^2}{4\pi\Phi}$
and $0 < \Phi^2 < \frac{d-3}{d-2}$.

Solution r_{+1} (unstable) exists in:
$RT_0 < RT$ and $0 < \Phi^2 < 1$.

In $d = 5$, solutions are roots of a quadratic.



$\Phi = 0.2$ and $RT = 0.4$



$\Phi = \{0.001; 0.2; 0.4; 0.6; \sim 0.7\}$

Thermodynamics of the ensemble

From the zero loop approximation, we have $Z = e^{-I_0}$.

To obtain the thermodynamics, we can use the relation $Z = e^{-\beta W}$, with $W = E - TS - \phi Q$ being the grand potential, and so $I_0 = \beta W$.

Through the derivatives of the grand potential, we get

$$S = \frac{A_+}{4}\,, \qquad Q = q\,,$$
$$p = \frac{d-3}{16\pi R\sqrt{f}}\left(\left(1-\sqrt{f}\right)^2 - \frac{\lambda q^2}{R^{2d-6}}\right)\,.$$

The energy given by $E = W + TS + \phi Q$ is

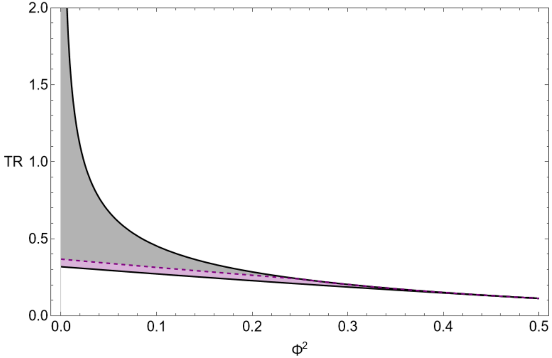
$$E = \frac{(d-2)\Omega R^{d-3}}{8\pi}(1-\sqrt{f})\,.$$

Stable if $C_{A,\phi} > 0$.

Phase transitions

Ensemble with two solutions: stable black hole or "hot flat space". The system prefers to be at the lowest value of W. We mimic "hot flat space" as a very small charged shell (without gravity).

$$W_{\text{shell}} = -\frac{(d-3)\Omega r_0^{d-3}}{2\left(1-\frac{r_0^{d-3}}{R^{d-3}}\right)}\phi^2 \, .$$



Conclusions and summary

- Two solutions arise from the grand canonical ensemble of a charged black hole, r_{+1} (small) and r_{+2} (large);
- r_{+2} is stable and only lives in a finite interval of temperatures, r_{+1} is unstable;
- Energy is the quasilocal energy, pressure as the one of the shell, entropy is the Bekenstein-Hawking;
- Stability is controlled by the heat capacity with constant ϕ and area (same as the case of a shell alone!);
- Hot flat space is more favorable until a maximum of r_+/R different than Buchdahl-Andréasson-Wright bound.