Constructing Accelerating NUT Black holes

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Plebanski-Demianski

The Plebanski-Demianski spacetime is the most general type D solution of Einstein-Maxwell equations and it represents a pair of charged rotating black holes, endowed with NUT charge, that accelerate from each other in opposite directions. [Plebanski-Demianski,1976]

$$ds^2 = \frac{1}{\Omega^2}\left(-\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta\right)d\varphi\right]^2 + \frac{\rho^2}{Q}dr^2 + \frac{\rho^2}{P}d\theta^2 + \frac{P}{\rho^2}\sin^2\theta\left[adt - \left(r^2 + (a+l)^2\right)d\varphi\right]^2\right)$$

where the functions are given by

$$\Omega = 1 - \frac{\alpha a}{a^2+l^2}r(l + a\cos\theta),$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2,$$

$$P(\theta) = \left(1 - \frac{\alpha a}{a^2+l^2}r_+(l + a\cos\theta)\right)\left(1 - \frac{\alpha a}{a^2+l^2}r_-(l + a\cos\theta)\right),$$

$$Q(r) = (r - r_+)(r - r_-)\left(1 + \alpha a\frac{a-l}{a^2+l^2}r\right)\left(1 - \alpha a\frac{a+l}{a^2+l^2}r\right).$$

[Podolsky and Vratny, 2021, 2022]

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Seven physical parameters:

- m mass parameter
- a Kerr rotation
- l NUT
- e electric charge
- g magnetic charge
- α acceleration
- Λ cosmological constant

[Podolsky and Vratny, 2021, 2022]

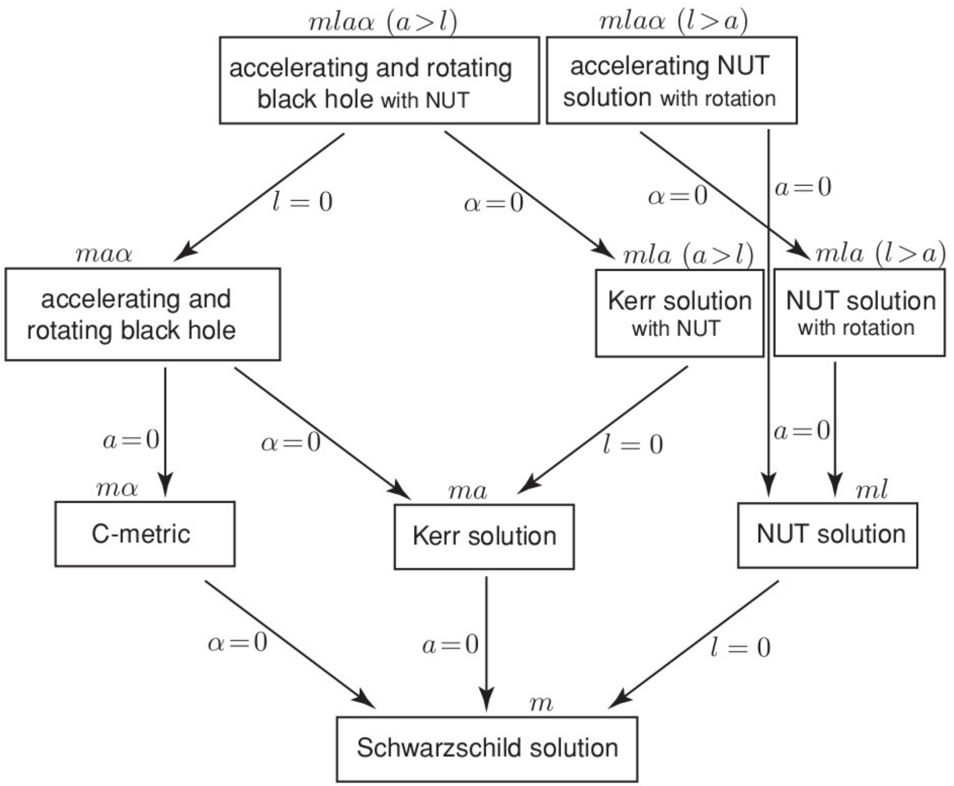
- When the rotation parameter a vanishes, the acceleration parameter α disappears.

- No accelerating-NUT solution!

- Until this point, an accelerating NUT solution without rotation has not been identified.

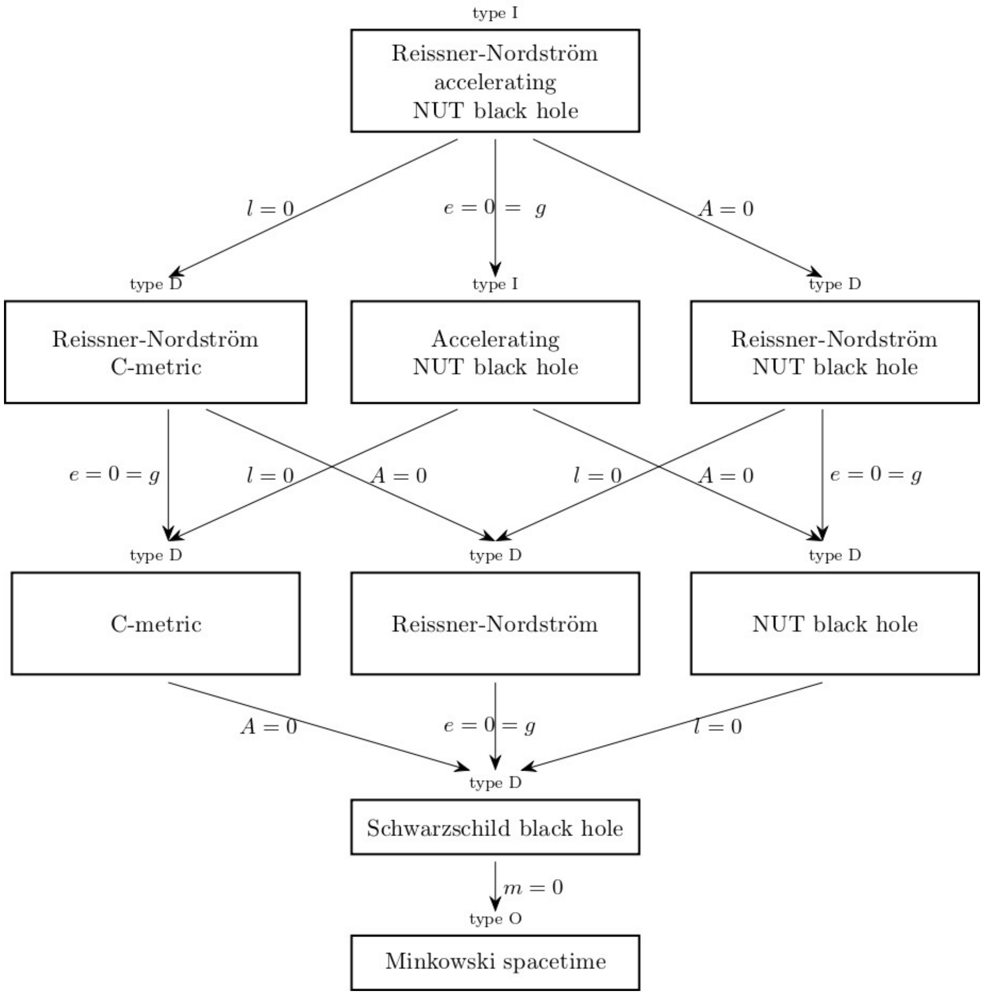
- If exist, should not belong to type D.

- Curvature singularity when $\rho^2 = 0$. If $|l| \leq |a|$, this occurs when r = 0 and $\cos\theta = -l/a$. On the other hand, if $|l| > |a|$, $\rho^2 = 0$ cannot be zero and the metric is non-singular.



[Griffiths and Podolsky, 2006]

Accelerating-NUT black holes



[JB and Cisterna, 2023]

- Uncharged case first found by Chng, Mann and Stelea, (2006) as an accelerating extension of the Zipoy-Voorhees-like family of solutions.

- The Weyl tensor is of algebraically general type I with four distinct principal null directions. [Podolsky and Vratny, 2020]

$$I^3 \neq 27J^2, \qquad I \equiv \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2 \qquad J \equiv \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}$$

- Could not been found within the large Plebanski-Demianski family.

- Previously known spacetimes are obtained in their classic form by simply setting the acceleration A, the NUT parameter l, or the mass m to zero.

How to construct?: Ernst Scheme

Ernst 1968: Einstein-Maxwell field equations for a general stationary and axisymmetric spacetime given by the Lewis-Weyl-Papapetrou (LWP) metric and a stationary and axially symmetric Maxwell field

$$ds^2 = -f(dt - \omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}\left(d\rho^2 + dz^2\right)\right],$$

$$A = A_t dt + A_\varphi d\varphi,$$

Electric ansatz

can be reduced to a couple of complex vectorial equations,

$$\left(\mathrm{Re}\,\mathcal{E} + |\Phi|^2\right)\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\left(\nabla\mathcal{E} + 2\Phi^*\nabla\Phi\right),$$

$$\left(\mathrm{Re}\,\mathcal{E} + |\Phi|^2\right)\nabla^2\Phi = \nabla\Phi\cdot\left(\nabla\mathcal{E} + 2\Phi^*\nabla\Phi\right),$$

where the complex Ernst potentials are defined as

$$\mathcal{E} = f - |\Phi|^2 + i\chi, \quad \Phi = A_t + i\tilde{A}_\varphi,$$

and

$$\hat{\varphi}\times\nabla\tilde{A}_\varphi = \rho^{-1} f\left(\nabla A_\varphi + \omega\nabla A_t\right),$$

$$\hat{\varphi}\times\nabla\chi = -\rho^{-1} f^2\nabla\omega - 2\hat{\varphi}\times\mathrm{Im}\left(\Phi^*\nabla\Phi\right).$$

The equations can be deduced from an effective action

$$I(\mathcal{E},\Phi)=\int dz\int d\rho\left[\frac{(\nabla\mathcal{E}+2\Phi^*\nabla\Phi)(\nabla\mathcal{E}^*+2\Phi\nabla\Phi^*)}{(\mathcal{E}+\mathcal{E}^*+2\Phi\Phi^*)^2}-\frac{\nabla\Phi\nabla\Phi^*}{\mathcal{E}+\mathcal{E}^*+2\Phi\Phi^*}\right]$$

The lagrangian enjoys a set of Lie point symmetries, which can be written as

$$\mathcal{E}=|\lambda|^2\mathcal{E}_0\,,\qquad \Phi=\lambda\Phi_0\,,$$

$$\mathcal{E}=\mathcal{E}_0+ib\,,\qquad \Phi=\Phi_0\,,$$

$$\mathcal{E}=\frac{\mathcal{E}_0}{1+i\jmath\mathcal{E}_0}\,,\qquad \Phi=\frac{\Phi_0}{1+i\jmath\mathcal{E}_0}\,,$$ Ehlers transformations

$$\mathcal{E}=\mathcal{E}_0-2\beta^*\Phi_0-|\beta|^2\,,\qquad \Phi=\Phi_0+\beta\,,$$

$$\mathcal{E}=\frac{\mathcal{E}_0}{1-2\alpha^*\Phi_0-|\alpha|^2\mathcal{E}_0}\,,\qquad \Phi=\frac{\alpha\mathcal{E}_0+\Phi_0}{1-2\alpha^*\Phi_0-|\alpha|^2\mathcal{E}_0}\,,$$ Harrison transformations

where b, j are real and α, β and λ are complex parameters.

$$
\begin{aligned}
ds^2 &= f(d\varphi - \omega dt)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 dt^2], \\
\Phi &= A_\varphi + i\tilde{A}_t.
\end{aligned}
$$

Magnetic ansatz

The effect of Harrison and Ehlers transformations on a given seed spacetime depends on which ansatz they act, electric or magnetic:

- Harrison
  - Electric ansatz → Electrify the seed solution
  - Magnetic ansatz → Embeds the seed in an EM universe
- Ehlers
  - Electric ansatz → Add NUT to the seed
  - Magnetic ansatz → Embeds the seed in a rotating background (swirling)

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Dressing the C-metric with NUT: Uncharged case

$$ds^2 = \frac{1}{\Omega(R,\theta)^2}\left[-Q(R)dt^2 + \frac{dR^2}{Q(R)} + \frac{R^2 d\theta^2}{P(\theta)} + R^2 P(\theta)\sin^2\theta d\varphi^2\right],$$

$$\Omega(R,\theta) = 1 + AR\cos\theta,$$
$$Q(R) = (1 - A^2R^2)\left(1 - \frac{2M}{R}\right),$$
$$P(\theta) = 1 + 2AM\cos\theta.$$

- Step 1: We compare the seed metric with the electric LWP ansatz. Then we recognize the functions to be:

$$f_0 = \frac{Q}{\Omega^2},$$
$$\rho = \frac{\sqrt{PQ}R\sin\theta}{\Omega^2},$$
$$\omega_0 = 0.$$

$$ds^2 = -f(dt - \omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}\left(d\rho^2 + dz^2\right)\right]$$

- Step 2: We write the seed Ernst potentials are

real, static seed ← $\mathcal{E}_0 = \frac{Q}{\Omega^2}, \quad \Phi_0 = 0.$ → the seed in uncharged

- Step 3: Transform the Ernst potential reads (the electromagnetic potential remains null)

$$\mathcal{E} = \frac{\mathcal{E}_0}{1 + ic\mathcal{E}_0} = \frac{Q/\Omega^2}{\Lambda}, \qquad \text{with} \qquad \Lambda(R,\theta) = 1 + ic\frac{Q}{\Omega^2}.$$

- Step 4: Read the transformed metric using the definitions of the Ernst potentials

$$f = \mathrm{Re}(\mathcal{E}) = \frac{Q/\Omega^2}{|\Lambda|^2} = \frac{f_0}{|\Lambda|^2},$$
$$\chi = \mathrm{Im}(\mathcal{E}) = -c\frac{Q^2/\Omega^4}{|\Lambda|^2}.$$

The last expresion provide us the existence of a rotating function given by

$$\hat{\varphi} \times \nabla\chi = -\rho^{-1} f^2 \nabla\omega - 2\hat{\varphi} \times \mathrm{Im}\left(\Phi^* \nabla\Phi\right) \longrightarrow \omega(R,\theta) = 2c\left(2M\cos\theta + \frac{AP(\theta)R^2\sin^2\theta}{\Omega(R,\theta)^2}\right).$$

- Step 5: Taking into account the functions f and ω the transformed metric takes the form

$$ds^2 = \frac{1}{\Omega(R,\theta)^2}\left[-\frac{Q(R)}{|\Lambda(R,\theta)|^2}\left[dt - 2c\left(2M\cos\theta + \frac{AP(\theta)R^2}{\Omega(R,\theta)^2}\sin^2\theta\right)d\varphi\right]^2 + |\Lambda(R,\theta)|^2\left(R^2P(\theta)\sin^2\theta d\varphi^2 + \frac{dR^2}{Q(R)} + R^2\frac{d\theta^2}{P(\theta)}\right)\right].$$

- Considering the reparametrization and change of coordinates of the form

$$m = \sqrt{M^2 - l^2}, \qquad r_\pm \equiv m \pm \sqrt{m^2 + l^2}, \qquad c = \frac{l}{r_+},$$

$$R \to (r - r_-), \qquad t \to \frac{(r_+ - r_-)}{r_+}\tau.$$

Accelerating-NUT black holes

- We find the final form of the spacetime to be

$$ds^2 = \frac{1}{\Omega(r,\theta)^2}\left[-\frac{\mathcal{Q}(r)}{\mathcal{R}^2(r,\theta)}\left[d\tau - 2l\left(\cos\theta + A\mathcal{T}(r,\theta)\sin^2\theta\right)d\varphi\right]^2 + \mathcal{R}^2(r,\theta)\left(\mathcal{P}(\theta)\sin^2\theta d\varphi^2 + \frac{dr^2}{\mathcal{Q}(r)} + \frac{d\theta^2}{\mathcal{P}(\theta)}\right)\right],$$

where

$$\Omega(r,\theta) = 1 + A\left(r - r_-\right)\cos\theta,$$

$$\mathcal{P}(\theta) = 1 + A\left(r_+ - r_-\right)\cos\theta,$$

$$\mathcal{Q}(r) = \left(1 - A^2\left(r - r_-\right)^2\right)\left(r - r_-\right)\left(r - r_+\right),$$

$$\mathcal{T}(r,\theta) = \frac{P(\theta)\left(r - r_-\right)^2}{\left(r_+ - r_-\right)\Omega(r,\theta)^2},$$

$$\mathcal{R}^2(r,\theta) = \frac{1}{r_+^2 + l^2}\left[r_+^2\left(r - r_-\right)^2 + l^2\frac{\left[1 - A^2\left(r - r_-\right)^2\right]^2}{\Omega(r,\theta)^4}\left(r - r_+\right)^2\right].$$

$$r_+ \equiv m + \sqrt{m^2 + l^2}\,,$$

$$r_- \equiv m - \sqrt{m^2 + l^2}\,,$$

$$m = \sqrt{M^2 - l^2}\,.$$

What if we consider electromagnetic charges into the seed?

Reissner-Nordstrom accelerating NUT black hole

- EM charges restrict the usefulness of the standard electric Ehlers transformation as a mechanism of endorsing NUT to a given spacetime.

- The Ehlers map rotates not only the mass, producing the appearance of the NUT parameter, but also the gauge vector generating its misalignment.

- To add NUT onto the Kerr-Newman solution the Ehlers map requieres to be complemented with a duality transformation affecting the electromagnetic potential.

- If we consider our seed as a charged C-metric

$$ds^2 = \frac{1}{\Omega(R,\theta)^2}\left[-Q(R)dt^2 + \frac{dR^2}{Q(R)} + \frac{R^2 d\theta^2}{P(\theta)} + R^2 P(\theta)\sin^2\theta d\varphi^2\right],$$

$$Q(R) = (1 - A^2R^2)\left(1 - \frac{2M}{R} + \frac{e^2+g^2}{R^2}\right),$$
$$P(\theta) = 1 + 2AM\cos\theta + A^2\left(e^2+g^2\right)\cos^2\theta,$$
$$A = -\frac{e}{R}dt + g\cos\theta d\phi.$$

- We identified the Ernst potentials (ε and Φ due the presence of EM charges) and following the procedure, we finally obtain the same line element of the uncharged case but now the functions:

$$\Omega(r,\theta) = 1 + A(r - r_-)\cos\theta,$$
$$\mathcal{P}(\theta) = 1 + A(r_+ - r_-)\cos\theta + A^2(e^2+g^2)\cos^2\theta,$$
$$\mathcal{Q}(r) = (1 - A^2(r-r_-)^2)(r^2 - 2mr - l^2 + e^2 + g^2),$$
$$\mathcal{T}(r,\theta) = \frac{\mathcal{P}(\theta)(r-r_-)^2}{(r_+ - r_-)\Omega(r,\theta)^2} - \frac{e^2+g^2}{r_+ - r_-},$$
$$\mathcal{R}^2(r,\theta) = \frac{1}{r_+^2 + l^2}\left[r_+^2(r-r_-)^2 + l^2\frac{\left(\mathcal{Q}(r) - (e^2+g^2)\Omega(r,\theta)^2\right)^2}{\Omega(r,\theta)^4(r-r_-)^2}\right],$$
$$\omega(r,\theta) = \frac{2l(r_+ - r_-)}{r_+}\left(\cos\theta + A\mathcal{T}(r,\theta)\sin^2\theta\right),$$

$$A_\tau(r,\theta) = \frac{(r-r_-)^2}{\mathcal{R}^2(r,\theta)}\left(-\frac{e}{(r-r_-)} - gl\frac{(\mathcal{Q}(r) - (e^2+g^2)\Omega(r,\theta)^2)}{r_+\Omega(r,\theta)^2(r-r_-)^3}\right),$$
$$A_\varphi(r,\theta) = l\frac{e\mathcal{P}(\theta)\sin^2\theta}{r_+\cos\theta\Omega(r,\theta)^2} + g\cos\theta - l\frac{e}{r_+\cos\theta} - \omega(r,\theta)A_\tau(r,\theta).$$

- When $A=0$, it is direct to obtain the NUT Reissner-Nordstrom metric for the line element.

- The vector potential remains rotated [$g=0$].

$$A_\varphi = -ce\cos\theta - \omega A_\tau$$

- An extra duality rotation fixes the problem

$$\Phi \to \Phi' = e^{i\beta}\Phi.$$

- In the case of A=0, β provides

$$\bar{A}_\varphi = -ce\cos\theta\cos\beta + e\cos\theta\sin\beta - \omega\bar{A}_t.$$

- In fact, β goes to a constant, $\arctan(c)$, only in the non-accelerating limit

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Final remarks

- One way to avoid the missaligment of the gauge fields when $A=0$ is apply the enhanced Ehlers transformations [Astorino, 2019, 2023]

$$\mathcal{E} = \frac{\mathcal{E}_0 + ib}{1 + ib\mathcal{E}_0}, \qquad \Phi = \frac{\Phi_0(1+ib)}{1 + ib\mathcal{E}_0}.$$

- Charging via Harrison transformations? New Type I black holes [Work in progress].

- Transformations change the classification type of the solution. Why?

- Mixing transformations: Type I non-trivial background (Melvin-Swirling, Swirling-NUT, charged Swirling) [Work in progress].

Thanks!