Black hole entropy and noncommutativity

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Black hole entropy

- S_{BH} was first introduced by Bekenstein as a useful analogy:

Thought experiment

Drop a particle into a black hole. The entropy of the exterior spacetime will drop, violating 2nd law of thermodynamics. Introduce $S_{BH} \propto A_{BH}$ so that

$$\partial_t \left(S_{\text{Ext}} + S_{BH} \right) \geq 0 \tag{1}$$

- S_{BH} was given a precise meaning after Hawking's proofs of black hole thermal radiation and laws of BH thermodynamics.

Noncommutative geometry

Using the Drinfeld twist

$$\mathcal{F} = e^{\frac{-i}{2}\Theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \tag{2}$$

and Hopf algebra formalism we can deform the geometric structure of GR, but also the pointwise multiplicative algebra of functions.

Deforming the product in the algebra of functions over the spacetime introduces noncommutativity between spacetime coordinates

$$[x^\mu, x^\nu]_\Theta = i\Theta^{\mu\nu} \ . \tag{3}$$

$$\forall f, g \in C^\infty(\mathcal{M}) : f \star g = \cdot(\mathcal{F} f \otimes g) \tag{4}$$

Brick wall - motivation

- It is known that any field theoretic calculation of BH entropy leads to UV divergences.
- Brick wall method, devised by 't Hooft is one way to approach regularization.

Consider a massless scalar field ϕ in the spacetime exterior to black hole. We can calculate the entropy of the exterior scalar field system using the statistical partition function

$$Z = \sum e^{-\beta E_i} \tag{5}$$

whose entropy will, due to entanglement, also correspond to BH entropy.

Brick wall

- Certainly, one way to find the partition function is to analytically find the spectrum of the scalar in the BH metric by solving its E.O.M.

$$\Box_g \phi = 0 \,, \tag{6}$$

where

$$g = \text{diag}\left(-f(r), \frac{1}{f(r)}, r^2\sin(\theta), r^2\right) \,. \tag{7}$$

- We can separate the angular and time coordinates and what remains is an equation of the form

$$R'' - \left[\frac{V^2}{\hbar^2} - \Delta\right] R = 0 \,. \tag{8}$$

- Approximatively, we can approach the problem by applying the WKB approximation to the radial part of the equation.

Brick wall

Apply the WKB approximation to the radial equation

$$R = \frac{c_0}{\sqrt{P(r)}} e^{\frac{i}{\hbar} \int^r P(x) dx} \tag{9}$$

and expand $P(r)$ into a power series in $\hbar^2$

$$P(r) = \sum_{n=0}^{\infty} \hbar^{2n} P_{2n}(r) \tag{10}$$

which yields

$$\begin{aligned} P_0(r) &= V(r) \\ P_2(r) &= \frac{3}{8P_0} \left(\frac{P_0'}{P_0} \right)^2 - \frac{P_0''}{4P_0} - \frac{\Delta}{2P_0} \\ P_4(r) &= \dots \end{aligned} \tag{11}$$

Brick wall

Imposing the Born-Sommerfeld quantization

$$\int_{r_H}^{\infty} P(r)dr = N(E, l, m, ...)\pi \tag{12}$$

gives the number of states

$$N(E) = \frac{1}{\pi}\int_{r_H+h}^{L} P(r)dr \tag{13}$$

where we have introduced the brick wall and IR regularization (h and L).

Brick wall

We calculate the free energy F

$$F = -\frac{1}{\beta}\int_0^\infty \frac{N(E)}{e^{\beta E}-1}dE \tag{14}$$

and substitute Hawking temperature

$$\beta = \frac{2\pi}{\hbar c}\frac{1}{\kappa} \ . \tag{15}$$

Finally, calculate the BH entropy

$$S = \beta^2 \frac{\partial^2 F}{\partial \beta^2} \ . \tag{16}$$

and set brick wall cutoff h to the appropriate value.

Noncommutative gravity

- We can use the Hopf algebra and Drinfeld twist formalism to deform the diffeomorphism algebra underlying GR (and also the scalar field theory).

This deforms the objects which define Einstein's equation

$$\begin{aligned} g_{ab} &\mapsto g^{\star}_{ab} \\ R_{ab} &\mapsto R^{\star}_{ab} \\ T_{ab} &\mapsto T^{\star}_{ab} \end{aligned} \tag{17}$$

$$R^{\star}_{ab} - \frac{1}{2} g^{\star}_{ab} \star R^{\star} + \Lambda g^{\star}_{ab} = 8\pi T^{\star}_{ab} \tag{18}$$

Noncommutative field theory

Using the same $\star$ product we can deform the scalar field theory in the exterior of the BH

$$S[\hat{\phi}] = \int \sqrt{-g^\star} \star \left(g^{\star\,\mu\nu} \star \left(D^\star_\mu \hat{\phi} \right)^+ \star D^\star_\mu \hat{\phi} \right) . \tag{19}$$

- The particular choice of the twist

$$\mathcal{F} = e^{\frac{-i}{2}\left(\partial_t \otimes \partial_\phi - \partial_\phi \otimes \partial_t\right)} \tag{20}$$

leaves gravity undeformed in the cases of Reissner–Nordström and charged BTZ black holes and only deforms the scalar field theory.

Noncommutative correction

- The deformed scalar field $\hat{\phi}$ can be expanded (Seiberg-Witten map) in an infinite series in commutative fields (ϕ and A_μ)

We obtain corrections to ϕ's E.O.M. from $\hat{\phi}$'s action

$$(\Box_g + \mathcal{D}(\Theta))\,\phi = 0 \tag{21}$$

- Applying the explained brick wall procedure, we can calculate corrections to the Bekenstein-Hawking law of entropy.

Results

The described procedure applied to the Reissner–Nordström BH yields the following type of correction terms

$$S = S_{BH} + \Theta^2 \mathcal{V}(A) + \Theta^2 \mathcal{W}(A) ln\left(\frac{A}{l_P^2}\right) , \tag{22}$$

where $\mathcal{V}(A)$ and $\mathcal{W}(A)$ are functions in A.

The described procedure applied to the charged BTZ BH yields the following type of correction terms

$$S = S_{BH} + \sum_{n=0}^{\infty} \mathcal{V}_n(A) ln^n\left(\frac{A}{l_P^2}\right) + \Theta^2 \sum_{n=0}^{\infty} \mathcal{W}_n(A) ln^n\left(\frac{A}{l_P^2}\right) , \tag{23}$$

where $\mathcal{V}_n(A)$ and $\mathcal{W}_n(A)$ are functions in A.

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This presentation is based on the following papers :

1. T. Jurić, F.P. :
   Noncommutative Correction to the Entropy of Charged BTZ Black Hole, arXiv:2212.06496
2. K.S. Gupta, T. Jurić, A. Samsarov, I. Smolić :
   Noncommutativity and logarithmic correction to the black hole entropy, arXiv:2209.07168
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